

Fractional Operators Approach and Fractional Boundary Conditions

Eldar Veliev¹, Turab Ahmedov² and Maksym Ivakhnychenko³

¹State Research and Design Institute of Basis Chemistry

²Institute of Mathematics NAS of Azerbaijan

³Institute of Radiophysics and Electronics NAS of Ukraine

^{1,3}Ukraine

²Azerbaijan

1. Introduction

Tools of fractional calculus including fractional operators and transforms have been utilized in physics by many authors (Hilfer, 2000). Fractional operators defined as fractionalizations of some commonly used operators allow describing of intermediate states. For example, fractional derivatives and integrals (Oldham & Spanier, 1974; Samko et al., 1993) are generalizations of derivative and integral. Fractional curl operator defined in (Engheta, 1998) is a fractionalized analogue of conventional curl operator used in many equations of mathematical physics. A fractionalized operator generalizes the original operator. The idea to use fractional operators in electromagnetic problems was formulated by N. Engheta (Engheta, 2000) and named "fractional paradigm in electromagnetic theory".

Our purpose is to find possible applications of the use of fractional operators in the problems of electromagnetic wave diffraction. In this paper two-dimensional problems of diffraction by infinitely thin surfaces are considered: a strip, a half-plane and a strip resonator (Fig.1). Assume that an incident field is an E-polarized plane wave, described by the function

$$\vec{E}^i = \vec{z}E_z^i(x, y) = \vec{z}e^{-ik(x \cos \theta + y \sin \theta)}, \quad (1)$$

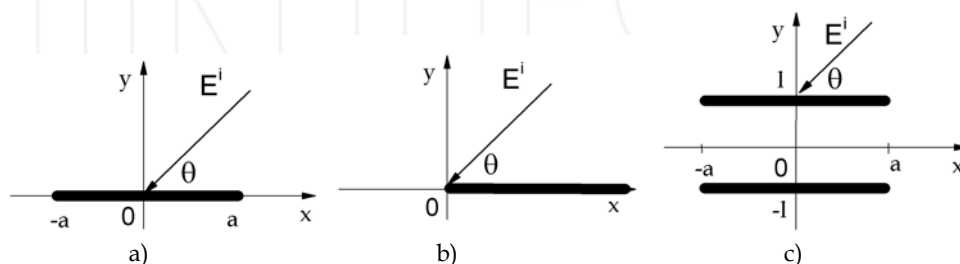


Fig. 1. Geometry of the diffraction problems: a) strip, b) half-plane, c) two parallel strips.

where θ is the incidence angle, $k = \frac{2\pi}{\lambda}$ is the wavenumber. Here, the time dependence is assumed to be $e^{-i\omega t}$ and omitted throughout the paper. There are three structures considered in this paper:

- a strip located in the plane $y=0$ ($x \in [-a, a]$) infinite along the axis z (Fig. 1a);
- a half-plane ($y=0, x \geq 0$) (Fig. 1b);
- two parallel strips infinite along the axis z (a strip resonator). The first strip is located at $y=l, x \in [-a, a]$, and the second one is at $y=-l, x \in [-a, a]$ (Fig. 1c).

One may ask what new features are that the fractional operators can bring to the theory of diffraction. The concept of intermediate states, obtained with the aid of fractional derivatives and integrals, yields to various generalizations of commonly used models in electrodynamics such as:

- Intermediate waves. For instance, intermediate waves between plane and cylindrical waves (Engheta, 1996, 1999) can be obtained using fractional integral of scalar Green's function:

$$G^\alpha(x, y; k) \equiv \frac{1}{2} (-{}_{-\infty}D_y^{-\alpha} G_2(x, y; k) - {}_{-\infty}D_{-y}^{-\alpha} G_2(x, y; k)), \quad 0 \leq \alpha \leq 1,$$

where G_2 is two-dimensional Green's function of the free space. G^α describes an intermediate case between one- and two-dimensional Green's functions and have the following behavior in the far-zone (Engheta, 1999):

$$G^\alpha \sim \frac{i}{4\pi} \cos\left(\frac{\pi\alpha}{2}\right) (k \sin|\varphi|)^{-\alpha} \sqrt{\frac{2\pi}{k\rho}} e^{ik\rho - i\pi/4} + \frac{i}{2k^\alpha} \Gamma(\alpha) \frac{e^{ik|x|}}{k|y|^{1-\alpha}}, \quad k\rho = k\sqrt{x^2 + y^2} \rightarrow \infty, \quad \varphi \neq 0.$$

This function consists of two waves: a cylindrical wave and a non-uniform plane wave propagating in the x direction and behaving with y as $|y|^{\alpha-1}$.

- Fractional Green's function G^α defined as a fractional derivative (integral) of the ordinary Green's function of the free space - $G^\alpha \equiv {}_{-\infty}D_{ky}^\alpha G$. α denotes the fractional order and varies from 0 to 1 ($0 \leq \alpha \leq 1$). In two-dimensional case G^α is expressed as

$$G^\alpha(x - x', y - y') = -\frac{i}{4} D_{ky}^\alpha H_0^{(1)}(k\sqrt{(x - x')^2 + (y - y')^2}). \quad (2)$$

- Fractional Green's theorem which involves fractional derivatives of ordinary Green's function and fractional derivatives of the considered function on a boundary of a domain (Veliev & Engheta, 2003). The corresponding equations will be presented later in this paper.
- Fractional boundary conditions (FBC) defined via fractional derivatives of the tangential electric field components $U(x, y)$. For an infinitely thin boundary S located in the plane $y = d$, FBC are defined as

$$D_y^\alpha U(x, y)|_{y \in S} = 0, \quad y \rightarrow \pm d.$$

The order of the fractional derivative α is assumed to be between 0 and 1. Fractional derivative D^α is applied along the direction normal to the surface S . Fractional boundary conditions describe an intermediate boundary between the perfect electric conductor (PEC) and the perfect magnetic conductor (PMC), obtained from FBC if the fractional order equals to 0 and 1, respectively.

We will use the symbol $D_y^\alpha f$ to denote operator of fractional derivative or integral ${}_{-\infty}D_y^\alpha f$, which is defined by the integral of Riemann-Liouville on semi-infinite interval (Samko et al., 1993):

$$({}_{-\infty}D_x^\alpha f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\infty}^x \frac{f(t)dt}{(x-t)^\alpha}, \quad 0 < \alpha < 1,$$

where $\Gamma(1-\alpha)$ is Gamma function.

This paper is devoted to the problems of diffraction by a strip, a strip resonator and a half-plane characterized with fractional boundary conditions with $0 \leq \alpha \leq 1$ expressed as

$$D_{ky}^\alpha E_z(x, y) = 0, \quad y \rightarrow \pm 0, \quad x \in L,$$

where $L = (-a, a)$ for a strip and $L = (0, \infty)$ for a half-plane. For convenience, fractional derivative is applied with respect to dimensionless variable ky . The function $E_z(x, y)$ denotes z-component of the total electric field, $E_z(x, y) = E_z^i + E_z^s$, that is the sum of the incident plane wave $E_z^i(x, y)$ and the scattered wave $E_z^s(x, y)$.

In case of a strip resonator we have two equations to impose fractional boundary conditions:

$$D_{ky}^\alpha E_z(x, y) = 0, \quad y \rightarrow l \pm 0, \quad x \in (-a, a),$$

$$D_{ky}^\alpha E_z(x, y) = 0, \quad y \rightarrow -l \pm 0, \quad x \in (-a, a).$$

From the one hand, introduction of new boundary conditions should describe a new physical boundary world, and from the other hand they must allow to build an effective computational algorithm to solve the stated problems with a desired accuracy. Simple mathematical description of the scattering properties of surfaces is a common problem in modeling in diffraction theory.

One of the well-studied boundaries, which can be treated as an intermediate state between PEC and PMC, is an impedance boundary defined by the equation

$$\vec{n} \times \vec{E}(\vec{r}) = \eta \vec{n} \times (\vec{n} \times \vec{H}(\vec{r})), \quad \vec{r} \rightarrow S,$$

where \vec{n} is the normal to the surface S . The value of the impedance η varies from 0 for PEC to $i\infty$ for PMC.

There are many papers devoted to diffraction by impedance boundaries. Impedance boundary conditions (IBC) have been used for the modeling of the scattering properties of good conductors, gratings, etc. In each case there are formulas to define the value of the impedance as a function of material parameters. IBC are approximate BC and therefore they have limitations in usage and cannot describe all diversity of boundaries.

Further approximation of IBC can be made with the aid of derivatives of higher but integer orders or generalized boundary conditions (Hope & Rahmat-Samii, 1995; Senior & Volakis, 1995). A general methodology to obtain exact IBC of higher order in spectral domain is presented in (Hope & Rahmat-Samii, 1995), where flat covers (and also surfaces with curvature) consisting of homogeneous materials with an arbitrary (linear, bi-anisotropic) constitutive equations. It is possible to obtain exact IBC in the spectral domain that can be often done in an analytical form very often. However, it is not always possible to get IBC in the spatial domain in an exact form. That is why it is necessary to approximate IBC in the spectral domain in order to apply inverse Fourier transform.

Another boundary condition that generalizes the perfect boundaries like PEC and PMC was introduced in (Lindell & Sihvola, 2005a). The corresponding surface was named perfect electromagnetic conductor (PEMC) and the mentioned condition is defined as

$$\vec{H} + M\vec{E} = 0.$$

For $M=0$, PEMC defines a PEC boundary and for $M=\infty$ we get a PMC. The physical model of PEMC boundary was proposed in (Lindell & Sihvola, 2005b) where it was shown that the PEMC condition can simulate reflection from an anisotropic layer for the normal incidence of the plane wave. Diffraction by a PEMC boundary has not been considered yet. Further generalization of PEMC can be made using concept of the generalized soft-and-hard surface (GSHS) (Haninnen et al., 2006):

$$\vec{a} \cdot \vec{E} = 0, \vec{b} \cdot \vec{H} = 0,$$

where \vec{a} , \vec{b} are complex vectors that satisfy equations $\vec{n} \cdot \vec{a} = \vec{n} \cdot \vec{b} = 0$ and $\vec{a} \cdot \vec{b} = 1$. GSHS can transform an incident plane wave with any given polarization into any other polarization of the reflected plane wave if the vectors \vec{a} , \vec{b} are chosen appropriately (Haninnen et al., 2006).

Fractional boundary conditions (FBC) can be compared with impedance boundary conditions (IBC). First of all FBC are intermediate between PEC and PMC as well as IBC. The value of fractional order $\alpha=0$ ($\alpha=1$) corresponds to the value of impedance $\eta=0$ ($\eta=i\infty$), respectively. For other values of $0 < \alpha < 1$ the deeper analysis is needed.

Physical analysis of the strip with FBC shows that the induced surface currents behave similarly to the currents on an impedance strip. Due to specific properties the strip with FBC is compared with the well-known impedance strip. It can be shown that for a wide range of input parameters the "fractional strip" behaves similarly to the impedance strip if the fractional order is chosen appropriately (Veliev et al., 2008b). The proposed method used for a "fractional strip" has some advantages over the known methods applied to the analysis of the wave scattering by an impedance strip.

The purpose of this work is to build an effective analytic-numerical method to solve two-dimensional diffraction problems for the boundaries described by fractional boundary conditions with $\alpha \in [0,1]$. The method will be applied to two canonical scattering objects: a strip and a half plane. The method is based on presenting the scattered field via fractional Green's function,

$$E_z^S(x, y) \equiv \int_L f^{1-\alpha}(x') G^\alpha(x - x', y) dx',$$

where $f^{1-\alpha}(x)$ is the unknown function and $G^\alpha(x-x', y) = -\frac{i}{4} D_{ky}^\alpha H_0^{(1)}(k\sqrt{(x-x')^2 + y^2})$ is the fractional derivative of the Green's function defined by equation (2). This presentation leads to the following dual integral equations (DIE) with respect to the Fourier transform $F^{1-\alpha}(q) = \int_L f^{1-\alpha}(\xi) e^{-ikq\xi} d\xi$ of the function $f^{1-\alpha}(x)$

$$\begin{cases} \int_{-\infty}^{\infty} F^{1-\alpha}(q) e^{ikd_L \xi q} (1-q^2)^{\alpha-1/2} dq = -4\pi e^{i\pi/2(1-\alpha)} \sin^\alpha \theta e^{-ikd_L \xi \cos \theta}, & \xi \in L, \\ \int_{-\infty}^{\infty} F^{1-\alpha}(q) e^{ikd_L \xi q} dq = 0, & \xi \notin L, \end{cases}$$

where $d_L = a$ for $L = (-a, a)$, $d_L = 1$ for $L = (0, \infty)$.

In the case of a strip resonator, we obtain more complicated set of integral equations which will be presented later in this paper.

The method generalizes the known method used for the PEC and PMC strip and half plane. As will be shown later, this method allows obtaining a solution for the value $\alpha = 0.5$ in the explicit analytical form. For other values of $\alpha \in [0, 1]$ the scattering problems are reduced to solving of the infinite systems of linear algebraic equations (SLAE). In order to discretize the DIE the function $f^{1-\alpha}(x)$ is represented as a series in terms of orthogonal polynomials: Gegenbauer polynomials for the strip and Laguerre polynomials for the half-plane. These representations result in a special kind of the edge conditions for the fractional current density function $f^{1-\alpha}(x)$. The physical characteristics of the considered scattering objects can be found with any desired accuracy by solving SLAE.

2. Diffraction by a strip with fractional boundary conditions

Assume that an E-polarized plane wave is characterized with the function $\vec{E}^i = \vec{z} E_z^i(x, y) = \vec{z} e^{-ik(x \cos \theta + y \sin \theta)}$. The total field $\vec{E} = \vec{z} E_z(x, y)$ must satisfy fractional boundary conditions

$$D_{ky}^\alpha E_z(x, y) = 0, \quad y \rightarrow \pm 0, \quad x \in L, \quad (3)$$

where $L = (-a, a)$ for a strip. For convenience, fractional derivative ${}_{-\infty}D_{ky}^\alpha$ is applied with respect to a dimensionless variable ky . The function $E_z(x, y)$ denotes the z-component of the total electric field $E_z(x, y) = E_z^i + E_z^s$ that is the sum of the incident plane wave $E_z^i(x, y)$ and the scattered field $E_z^s(x, y)$. Solution to the diffraction by the screen $S = \{(x, y) : y = 0, -a < x < a\}$ is to be sought under the following conditions:

- The total field \vec{E} must satisfy the Helmholtz equation everywhere outside the screen

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2 \right) E_z(x, y) = 0. \quad (4)$$

- The scattered field $E_z^s(x, y)$ must satisfy Sommerfeld radiation condition at the infinity

$$\lim_{r \rightarrow \infty} \sqrt{r} \left(\frac{\partial E_z^s}{\partial r} - i E_z^s \right) = 0, \quad r = \sqrt{x^2 + y^2} \quad (5)$$

- The total field \vec{E} must satisfy the edge condition, i.e. the finiteness of energy in every local area near the edges of the screen (Honl et al., 1961).
- The total field $E_z(x, y)$ must satisfy the boundary conditions (3).

The method is based on representation of the scattered field with the aid of the fractional derivative of the Green's function:

$$E_z^s(x, y) \equiv \int_L f^{1-\alpha}(x') G^\alpha(x - x', y) dx'. \quad (6)$$

In (6), the function $f^{1-\alpha}(x)$ is the unknown function called the density of the fractional potential, and G^α is the fractional derivative of two-dimensional the Green's function of the free space defined by equation (2).

For the limit cases of the fractional order with $\alpha=0$ and $\alpha=1$ representation (6) corresponds to the single-layer and double-layer potentials commonly used to present the scattered fields in diffraction problems:

$$E_z^s(x, y) = \begin{cases} -\frac{i}{4} \int_{-a}^a f'(x') H_0^{(1)}(k\sqrt{(x-x')^2 + y^2}) dx', & \alpha = 0 \\ -\frac{i}{4} \int_{-a}^a f(x') \frac{\partial}{\partial y} H_0^{(1)}(k\sqrt{(x-x')^2 + y^2}) dx', & \alpha = 1 \end{cases}$$

More general representations (6) can be derived from the fractional Green's theorem (Veliev & Engheta, 2003) which generalizes the ordinary Green's theorem.

2.1 Fractional Green's theorem

Consider a function $\psi(\vec{r})$, which satisfies inhomogeneous scalar Helmholtz equation with the source density given by the function $\rho(\vec{r})$:

$$\Delta \psi(\vec{r}) + k^2 \psi(\vec{r}) = -4\pi \rho(\vec{r}). \quad (7)$$

Besides, define $G(\vec{r}, \vec{r}_0)$ as the Green's function of the Helmholtz equation:

$$\Delta G(\vec{r}, \vec{r}_0) + k^2 G(\vec{r}, \vec{r}_0) = -4\pi \delta(\vec{r} - \vec{r}_0). \quad (8)$$

Here, $\delta(\vec{r} - \vec{r}_0)$ is the three-dimensional Dirac delta function, \vec{r} and \vec{r}_0 are the position vectors for the observation and source points, respectively, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian, and k is a scalar constant. After applying fractional derivatives to equations (7) and (8) with respect to the x variable, multiplying the first equation with ${}_{-\infty}D_x^\nu G(\vec{r}, \vec{r}_0)$, and the second with ${}_{-\infty}D_x^\mu \psi(\vec{r})$, subtracting one from another, integrating this over all source

coordinates x_0, y_0, z_0 inside S , and finally using the Green's theorem, we obtain the following representation:

$${}_{-\infty}D_x^\beta \psi(\vec{r}) = \begin{cases} \int_V {}_{-\infty}D_{x_0}^{\beta-\nu} \rho(\vec{r}_0) \bullet {}_{-\infty}D_{x_0}^\nu G(\vec{r}, \vec{r}_0) dv_0 + \\ + \frac{1}{4\pi} \oint_S [{}_{-\infty}D_{x_0}^\nu G(\vec{r}, \vec{r}_0) \bullet \nabla_0 {}_{-\infty}D_{x_0}^{\beta-\nu} \psi(\vec{r}_0) - {}_{-\infty}D_{x_0}^{\beta-\nu} \psi(\vec{r}_0) \bullet \nabla_0 {}_{-\infty}D_{x_0}^\nu G(\vec{r}, \vec{r}_0)] \bullet ds_0, & \vec{r} \in V \\ 0, & \vec{r} \notin V \end{cases} \quad (9)$$

where $\mu + \nu = \beta$. Operator ∇_0 denotes the operator of gradient in respect of variable $\vec{r}_0(x_0, y_0, z_0)$. Here it was used the property of the fractional derivative of the Dirac delta function:

$$\int_V F(\vec{r}_0) {}_{-\infty}D_{x_0}^\nu \delta(\vec{r}_0 - \vec{r}) dv_0 = {}_{-\infty}D_x^\nu F(\vec{r}), \quad (10)$$

We use the uniform symbol ${}_{-\infty}D_x^\alpha$ (or D_x^α) to denote both fractional derivatives and fractional integrals, and it defines a fractional derivative for $0 < \alpha < 1$ and a fractional integral for $\alpha < 0$.

Equation (9) is a generalization of well-known Green's theorem for the case of fractional derivatives.

Consider some important particular cases, which can be obtained from (9).

In the case of excitation in a free space so that the volume V is the whole space, the surface integrals in (9) vanish, and we have:

$${}_{-\infty}D_x^\beta \psi(\vec{r}) = \int_V {}_{-\infty}D_{x_0}^{\beta-\nu} \rho(\vec{r}_0) \bullet {}_{-\infty}D_{x_0}^\nu G(\vec{r}, \vec{r}_0) dv_0. \quad (11)$$

Originally function $\psi(\vec{r})$ characterizes the field excited by the source with the volume density $\rho(\vec{r})$. From the other hand, for $\beta = 0$ representation (11) means that the field $\psi(\vec{r})$ is expressed through the distribution of fractional sources with density $D^{-\nu} \rho(r_0)$ inside the volume V and by using fractional integral of conventional Green's function $D^\nu G(r_0, r)$.

Assuming $\rho(\vec{r}) = 0$, we can obtain some other important representations:

$${}_{-\infty}D_x^\beta \psi(\vec{r}) = \begin{cases} \frac{1}{4\pi} \oint_S [{}_{-\infty}D_{x_0}^\beta G(\vec{r}, \vec{r}_0) \bullet \nabla_0 \psi(\vec{r}_0) - \psi(\vec{r}_0) \bullet \nabla_0 {}_{-\infty}D_{x_0}^\beta G(\vec{r}, \vec{r}_0)] ds_0, & \text{if } \nu = \beta, \mu = 0 \\ \frac{1}{4\pi} \oint_S [G(\vec{r}, \vec{r}_0) \bullet \nabla_0 {}_{-\infty}D_{x_0}^\beta \psi(\vec{r}_0) - {}_{-\infty}D_{x_0}^\beta \psi(\vec{r}_0) \bullet \nabla_0 G(\vec{r}, \vec{r}_0)] ds_0, & \text{if } \nu = 0 \end{cases} \quad (12)$$

From this representation we see that the fractional derivative of function $\psi(\vec{r})$ is expressed either via the value of the function and its first derivative at the boundary and the fractional derivatives of Green's function, or by the fractional derivatives of the function at the boundary and the usual Green's function.

If $\nu = -\mu$, i.e. $\beta = 0$, we obtain a representation for the function $\psi(\vec{r})$ itself:

$$\psi(\vec{r}) = \frac{1}{4\pi} \oint_S [-\infty D_{x_0}^{-\mu} G(\vec{r}, \vec{r}_0) \cdot \nabla_{0-\infty} D_{x_0}^{\mu} \psi(\vec{r}_0) - \infty D_{x_0}^{\mu} \psi(\vec{r}_0) \cdot \nabla_{0-\infty} D_{x_0}^{-\mu} G(\vec{r}, \vec{r}_0)] ds_0. \quad (13)$$

This expression means that the function $\psi(\vec{r})$ is represented through its fractional derivatives at the boundary and the fractional derivatives of Green's function. The equation (13) can be useful in scattering problems. If we have boundary conditions for the function $\psi(\vec{r})$ on the surface S as $\nabla_{0-\infty} D_{x_0}^{\mu} \psi(\vec{r}_0)|_{\vec{r}_0 \in S} = 0$ (or $-\infty D_{x_0}^{\mu} \psi(\vec{r}_0)|_{\vec{r}_0 \in S} = 0$) then one of the surface integrals in (13) vanishes and we get a simple presentation for $\psi(\vec{r})$. This fact will be used to present the scattered field in all diffraction problems considered in this paper (6). Equations (12), (13) generalize the Huygens principle in such a sense that the fractional derivative of the function $\psi(\vec{r})$, which characterizes a wave process, is presented as a superposition of waves radiated by elementary "fractional" sources distributed on the given surface. "Fractional" potentials, $\oint_S -\infty D_{x_0}^{\beta-\nu} \psi(\vec{r}_0) \cdot \nabla_{0-\infty} D_{x_0}^{\nu} G(\vec{r}, \vec{r}_0) \cdot ds_0$, $\oint_S -\infty D_{x_0}^{\nu} G(\vec{r}, \vec{r}_0) \cdot \nabla_{0-\infty} D_{x_0}^{\beta-\nu} \psi(\vec{r}_0) \cdot ds_0$, can be treated as a generalization of well-known single and double layer potentials.

2.2 Solution to integral equations

Substituting the expression (6) for $E_z(x, y)$ into fractional boundary conditions (3) we get the equation

$$\lim_{y \rightarrow 0} D_{ky}^{\alpha} \int_L f^{1-\alpha}(x') G^{\alpha}(x - x', y) dx' = -\lim_{y \rightarrow 0} D_{ky}^{\alpha} E_z^i(x, y), \quad (14)$$

It is convenient to use the Fourier transform of the fractional potential density $f^{1-\alpha}(x)$

$$F^{1-\alpha}(q) \equiv \int_{-\infty}^{\infty} \tilde{f}^{1-\alpha}(\xi) e^{-ikq\xi} d\xi = a \int_{-1}^1 f^{1-\alpha}(a\xi) e^{-ikq\xi} d\xi,$$

where a new function $\tilde{f}^{1-\alpha}(\xi)$ is introduced:

$$\tilde{f}^{1-\alpha}(\xi) \equiv af^{1-\alpha}(a\xi), \quad |\xi| < 1,$$

$$\tilde{f}^{1-\alpha}(\xi) \equiv 0, \quad |\xi| \geq 1.$$

Then the scattered field is expressed via the Fourier transform $F^{1-\alpha}(q)$ as

$$E_z^s(x, y) = -i \frac{e^{\pm i\pi\alpha/2}}{4\pi} \int_{-\infty}^{\infty} F^{1-\alpha}(q) e^{ik(xq + |y|\sqrt{1-q^2})} (1-q^2)^{(\alpha-1)/2} dq, \quad (15)$$

where the upper (lower) sign is chosen for $y > 0$ ($y < 0$). Here, in (15), the following representation for the fractional Green's function was used:

$$G^{\alpha}(x - x', y) = -i D_{ky}^{\alpha} H_0^{(1)}(k\sqrt{(x - x')^2 + y^2}) = -i \frac{e^{\text{sign}(y)i\pi\alpha/2}}{4\pi} \int_{-\infty}^{\infty} e^{ik((x-x')q + |y|\sqrt{1-q^2})} (1-q^2)^{(\alpha-1)/2} dq \quad (16)$$

It can be shown that the equation (14) can be reduced to dual integral equations (DIE)

$$\begin{cases} \int_{-\infty}^{\infty} F^{1-\alpha}(q) e^{ika\xi q} (1-q^2)^{\alpha-1/2} dq = -4\pi e^{i\pi/2(1-\alpha)} \sin^{\alpha} \theta e^{-ikd_L \xi \cos \theta}, & |\xi| < 1, \\ \int_{-\infty}^{\infty} F^{1-\alpha}(q) e^{ika\xi q} dq = 0, & |\xi| > 1, \end{cases} \quad (17)$$

For the limit cases of the fractional order $\alpha = 0$ and $\alpha = 1$ the equations (17) are reduced to the well known integral equations used for PEC and PMC strips (Honl et al., 1961; Veliev & Veremey, 1993; Veliev & Shestopalov, 1988; Uflyand, 1977), respectively. In this paper the method to solve DIE (17) is proposed for arbitrary value of $\alpha \in [0,1]$.

DIE (17) can be solved analytically for one special case of $\alpha = 0.5$. In this case we get the solutions for any value of k as

$$f^{0.5}(x) = -2ik \sin^{1/2} \theta e^{-ikx \cos \theta + i\pi/4}, \quad (18)$$

$$F^{0.5}(q) = -4ie^{i\pi/4} \sin^{1/2} \theta \frac{\sin ka(q + \cos \theta)}{q + \cos \theta}. \quad (19)$$

In the case of arbitrary α the solutions can be obtained numerically. First, we modify the equations (17). After multiplying by $e^{-ika\tau\xi}$ and integrating in ξ from -1 to 1, the first equation in (17) can be rewritten in the following form:

$$\int_{-\infty}^{\infty} F^{1-\alpha}(q) \frac{\sin ka(q - \tau)}{q - \tau} (1-q^2)^{\alpha-1/2} dq = -4\pi e^{i\pi/2(1-\alpha)} \sin^{\alpha} \theta \frac{\sin ka(\tau + \cos \theta)}{\tau + \cos \theta}. \quad (20)$$

In order to discretize this equation, we present the unknown function $\tilde{f}^{1-\alpha}(\xi)$ as a uniformly convergent series in terms of the orthogonal polynomials with corresponding weight functions which allow satisfying the edge conditions:

$$\tilde{f}^{1-\alpha}(\xi) = (1 - \xi^2)^{\alpha-1/2} \sum_{n=0}^{\infty} f_n^{\alpha} \frac{1}{\alpha} C_n^{\alpha}(\xi), \quad (21)$$

where $C_n^{\alpha}(x)$ are the Gegenbauer polynomials and f_n^{α} are the unknown coefficients. Gegenbauer polynomials can be treated as intermediate polynomials between Chebyshev polynomials of the first and second kind:

$$\lim_{\alpha \rightarrow 0} \frac{C_n^{\alpha}(\xi)}{\alpha} = \begin{cases} \frac{2}{n} T_n(\xi), & n \neq 0 \\ 1, & n = 0 \end{cases}, \quad \lim_{\alpha \rightarrow 1} \frac{C_n^{\alpha}(\xi)}{\alpha} = C_n^1(\xi) = U_n(\xi).$$

The Fourier transform $F^{1-\alpha}(q)$ is expressed as the series

$$F^{1-\alpha}(q) = \frac{2\pi}{\Gamma(\alpha+1)} \sum_{n=0}^{\infty} (-i)^n \frac{\Gamma(n+2\alpha)}{\Gamma(n+1)} \frac{J_{n+\alpha}(kaq)}{(2kaq)^{\alpha}} f_n^{\alpha}, \quad (22)$$

where $J_{n+\alpha}(kaq)$ is the Bessel function.

It must be noted that the edge conditions are chosen in the following form

$$\tilde{f}^{1-\alpha}(\xi) = O\left((1-\xi^2)^{\alpha-1/2}\right), \quad \xi \rightarrow \pm 1. \quad (23)$$

For special cases of $\alpha = 0$ and $\alpha = 1$ the edge conditions have the form as

$$\tilde{f}^{1-\alpha}(\xi) = \begin{cases} O\left((1-\xi^2)^{-1/2}\right), & \alpha = 0 \\ O\left((1-\xi^2)^{1/2}\right), & \alpha = 1 \end{cases}, \quad \xi \rightarrow \pm 1 \quad (24)$$

These are well-known Meixner edge conditions in diffraction problems (Honl et al., 1961). Substituting (22) into (17) and taking into account the properties of discontinuous integrals of Weber-Shafheitlin (Bateman & Erdelyi, 1953) and the following formula (Prudnikov et al., 1986)

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{J_{n+\nu}(\varepsilon q)}{q^{\nu}} \frac{\sin \varepsilon(q-\beta)}{q-\beta} dq = \frac{J_{n+\nu}(\varepsilon \beta)}{\beta^{\nu}}, \quad (25)$$

one can show that the homogenous equation in the set (17) is satisfied identically.

The first equation of (17) written in the form (20) can be reduced to an infinite system of linear algebraic equations (SLAE) with respect to the unknown coefficients f_n^{α} :

$$\sum_{n=0}^{\infty} (-i)^n \frac{\Gamma(n+2\alpha)}{\Gamma(n+1)} C_{mn}^{\alpha} f_n^{\alpha} = B_m^{\alpha}, \quad m = 0, 1, 2, \dots, \infty \quad (26)$$

where the matrix coefficients are expressed as

$$C_{mn}^{\alpha} = \int_{-\infty}^{\infty} J_{n+\alpha}(kaq) J_{m+\alpha}(kaq) \frac{(1-q^2)^{\alpha-1/2}}{q^{2\alpha}} dq,$$

$$B_m^{\alpha} = -2\Gamma(\alpha+1)(2ka)^{\alpha} e^{i\pi/2(1-\alpha)} \sin^{\alpha} \theta \frac{J_{m+\alpha}(ka \cos \theta)}{(\cos \theta)^{\alpha}}.$$

It can be shown that the SLAE (26) can be reduced to SLAE of the Fredholm type of the second kind (Veliev et al., 2008a). Then the coefficients f_n^{α} can be found with any desired accuracy (within the machine precision) using the truncation of SLAE. The fractional density $f^{1-\alpha}(x)$ is computed by using (21) and the scattered field (6) and other physical characteristics can be obtained as series in terms of the found coefficients f_n^{α} .

In order to solve the diffraction problem on a plane screen with fractional boundary conditions and obtain a convenient SLAE we applied several techniques. First of all, the fractional Green's theorem presented above allowed searching the unknown scattered field as a potential with the fractional Green's function. The order of the fractional Green's function is defined from the fractional order of the boundary conditions. In general, the fractional derivative of Green's function may have a complicated form, but we used the Fourier transform where application of the fractional derivative maps to a simple multiplication by $(iq)^{\alpha}$. Finally, utilization of the orthogonal Gegenbauer polynomials along with the specific form of the edge conditions

allowed to reduce integral equations to SLAE in a convenient form. One can compare the method presented for fractional boundary conditions with the known methods applied to solve diffraction by an impedance strip. The impedance strip requires to consider two unknown densities in presentation of the scattered field as a sum of single- and double-layer potentials. The usage of two unknown functions leads to more complicated SLAE in spite of the SLAE obtained for fractional boundary conditions.

2.3 Physical characteristics

We consider such electrodynamic characteristics of the scattered field as the radiation pattern (RP), monostatic radar cross-section (MRCS) and surface current densities depending on the coefficients f_n^α . The scattered field $E_z^s(x, y)$ in the far-zone $kr \rightarrow \infty$ in the cylindrical coordinate system (r, ϕ) , $x = r \cos \phi, y = r \sin \phi$, is expressed as

$$E_z^s(r, \phi) = \frac{i}{4\pi} (\pm i)^\alpha \int_{-\infty}^{+\infty} F^{1-\alpha}(\cos \beta) e^{ikr \cos(\phi \pm \beta)} \sin^\alpha \beta d\beta,$$

where the upper sign is chosen for $\phi \in [0, \pi]$, and the lower one when $\phi \in [\pi, 2\pi]$. Using the stationary phase method for $kr \rightarrow \infty$ we present $E_z^s(x, y)$ as

$$E_z^s(x, y) \approx A(kr) \Phi^\alpha(\phi), \quad kr \rightarrow \infty, \quad (27)$$

where

$$A(kr) = \sqrt{\frac{2}{\pi kr}} e^{ikr - i\pi/4}, \quad \Phi^\alpha(\phi) = -\frac{i}{4} (\pm i)^\alpha F^{1-\alpha}(\cos \phi) \sin^\alpha \phi.$$

The function $\Phi^\alpha(\phi)$ describes RP and can be expressed via the coefficients f_n^α as

$$\Phi^\alpha(\phi) = \frac{\pi i (\pm i)^\alpha}{2\Gamma(\alpha + 1)} \tan^\alpha \phi \sum_{n=0}^{\infty} (-i)^n f_n^\alpha \frac{\Gamma(n + 2\alpha)}{\Gamma(n + 1)} \frac{J_{n+\alpha}(ka \cos \phi)}{(2ka)^\alpha}.$$

In physical optics (PO) approximation ($ka \gg 1$) $\Phi^\alpha(\phi)$ has a simpler form. Using the following formula

$$\lim_{ka \rightarrow \infty} \frac{\sin ka(\alpha - \beta)}{\alpha - \beta} = \pi \delta(\alpha - \beta), \quad (28)$$

in IE (20) we get the following expressions for $F^\alpha(q)$ and $\Phi^\alpha(\phi)$:

$$F^{1-\alpha}(q) \approx -4i^\alpha \frac{\sin^{1-\alpha} \theta}{(1 - q^2)^{(1-2\alpha)/2}} \frac{\sin ka(q - \cos \theta)}{q - \cos \theta},$$

$$\Phi^\alpha(\phi) \approx (\mp 1)^\alpha \sin \phi \left(\frac{\sin \theta}{\sin \phi} \right)^\alpha \frac{\sin ka(\cos \phi + \cos \theta)}{\cos \phi + \cos \theta}.$$

In the special case of $\alpha = 0.5$ and arbitrary value of ka we get an analytical expression for the RP

$$\Phi^{0.5}(\varphi) = (\mp 1)^{1/2} \sqrt{\sin \varphi \sin \theta} \frac{\sin ka(\cos \varphi + \cos \theta)}{\cos \varphi + \cos \theta}.$$

Bi-static radar cross section (BRCS) is expressed from RP $\Phi(\varphi)$ as $\frac{\sigma_{2d}}{\lambda}(\varphi) = \frac{2}{\pi} |\Phi(\varphi)|^2$. MRCS

$$\sigma_{2D}^{mono} \text{ is defined as } \sigma_{2D}^{mono} = \frac{\sigma_{2d}}{\lambda}(\theta) = \frac{2}{\pi} |\Phi(\theta)|^2.$$

We have the following representations in PO approximation

$$\frac{\sigma_{2d}}{\lambda} = \frac{2}{\pi} \sin^2 \varphi \left(\frac{\sin \theta}{\sin \varphi} \right)^{2\alpha} \left\{ \frac{\sin ka(\cos \varphi + \cos \theta)}{\cos \varphi + \cos \theta} \right\}^2, \quad ka \gg 1,$$

$$\sigma_{2D}^{mono} = \frac{2}{\pi} \sin^2 \theta \left\{ \frac{\sin ka(2 \cos \theta)}{2 \cos \theta} \right\}^2, \quad ka \gg 1.$$

It must be noted that the density function $f^{1-\alpha}(x)$ in the integral (6) does not describe the density of physical surface currents on the strip for $0 < \alpha < 1$. The function $f^{1-\alpha}(x)$ is defined as the discontinuity of fractional derivatives of E-field at the plane $y = 0$:

$$f^{1-\alpha}(x) = {}_{-\infty}D_{ky}^{1-\alpha} E_z(x, y)|_{y=+0} - {}_{-\infty}D_{ky}^{1-\alpha} E_z(x, y)|_{y=-0}, \quad x \in (-a, a). \quad (29)$$

For the limit cases of $\alpha = 0$ and $\alpha = 1$ the equation (29) is reduced to well-known presentations for electric and magnetic surface currents, respectively, i.e.

$$f^{1-\alpha}(x) = \begin{cases} \frac{\partial E_z(x, y)}{\partial y}|_{y=+0} - \frac{\partial E_z(x, y)}{\partial y}|_{y=-0} = H_x(x, +0) - H_x(x, -0), & \alpha = 0 \\ E_z(x, +0) - E_z(x, -0), & \alpha = 1 \end{cases}$$

In order to obtain physical surface currents from $f^{1-\alpha}(x)$ we have to apply additional integration. In case of E-polarized incident plane wave we have the following induced currents on a strip: electric current $\vec{j}^{\alpha(e)} = \vec{z} j_z^{\alpha(e)}$ and magnetic current $\vec{j}^{\alpha(m)} = \vec{x} j_x^{\alpha(m)}$ expressed from $f^{1-\alpha}(x)$ as

$$j_z^{\alpha(e)}(x) = -2i \cos\left(\frac{\pi\alpha}{2}\right) \frac{i}{4\pi} \int_{-\infty}^{+\infty} F^{1-\alpha}(q) e^{ikax} (1-q^2)^{\alpha/2} dq,$$

$$j_x^{\alpha(m)}(x) = -2 \sin\left(\frac{\pi\alpha}{2}\right) \frac{i}{4\pi} \int_{-\infty}^{+\infty} F^{1-\alpha}(q) e^{ikax} (1-q^2)^{\alpha/2-1/2} dq.$$

The detailed analysis of the scattering properties of the strip with fractional boundary conditions one can find in papers (Veliev et al., 2008a; Veliev et al., 2008b).

2.4 H-polarization

In the case of the H -polarized incident plane wave $\vec{H}^i(0,0,H_z^i)$, where $H_z^i(x,y) = e^{-ik(x\cos\theta+y\sin\theta)}$, the method proposed above can be applied as well. We define fractional boundary conditions as

$$D_{ky}^{1-\alpha} H_z(x,y)|_{y \rightarrow \pm 0} = D_{ky}^{1-\alpha} [H_z^i(x,y) + H_z^s(x,y)]|_{y \rightarrow \pm 0} = 0, \quad x \in (-a, a).$$

The case of $\alpha = 0$ corresponds to diffraction of the H -polarized plane wave on a PEC strip, while the case of $\alpha = 1$ describes diffraction of the H -polarized plane wave on a PMC strip. As before, we represent the scattered field via the fractional Green's function

$$H_z^s(x,y) \equiv \int_{-a}^a f^\alpha(x') G^{1-\alpha}(x-x',y) dx'.$$

After substituting (18) into fractional boundary conditions (19) we get the equation

$$\lim_{y \rightarrow 0} D_{ky}^{1-\alpha} \int_{-a}^a f^\alpha(x') G^{1-\alpha}(x-x',y) dx' = -\lim_{y \rightarrow 0} D_{ky}^{1-\alpha} H_z^i(x,y).$$

This equation can be solved by repeating all steps of the E -polarization case after changing α to $1-\alpha$.

3. Diffraction by a half-plane with fractional boundary conditions

Another problem studied in this paper is the diffraction by a half-plane with fractional boundary conditions. The method introduced to solve the dual integral equation (DIE) for a finite object (a strip) will be modified to solve DIE for semi-infinite scatterers such as half-plane. There are many papers devoted to the classical problem of diffraction by a half-plane. The method to solve the scattering problem for a perfectly conducting half-plane is presented in (Honl et al., 1961). Usually, it is solved using Wiener-Hopf method. The first application of the method to a PEC half-plane can be referred to the papers of Copson (Copson, 1946) and independently to papers of Carlson and Heins (Carlson & Heins, 1947). In 1952 Senior first applied Wiener-Hopf method to the diffraction by an impedance half-plane (Senior, 1952) and later oblique incidence was considered (Senior, 1959). Diffraction by a resistive and conductive half-plane and also by various types of junctions is analyzed in details in (Senior & Volakis, 1995). We propose a new approach for the rigorous analysis of the considered problem which generalizes the results of (Veliev, 1999) obtained for the PEC boundaries and includes them as special cases.

Let an E -polarized plane wave $E_z^i(x,y) = e^{-ik(x\cos\theta+y\sin\theta)}$ (1) be scattered by a half-plane ($y = 0, x > 0$). The total field $E_z = E_z^i + E_z^s$ must satisfy fractional boundary conditions

$$D_{ky}^\alpha E_z(x,y) = 0, \quad y \rightarrow \pm 0, \quad x > 0, \quad (30)$$

and Meixner's edge conditions must be satisfied for $x \rightarrow 0$.

Following the idea used for the analysis of diffraction by a strip we represent the scattered field using the fractional Green's function

$$E_z^s(x, y) \equiv \int_0^\infty f^{1-\alpha}(x') G^\alpha(x - x', y) dx', \quad (31)$$

where $f^{1-\alpha}(x)$ is the unknown function, G^α is the fractional Green's function (2). After substituting the representation (31) into fractional boundary conditions (30) we get the equation

$$-\frac{i}{4} \lim_{y \rightarrow 0} D_{ky}^{2\alpha} \int_0^\infty f^{1-\alpha}(x') H_0^{(1)} \left(k \sqrt{(x-x')^2 + y^2} \right) dx' = -\lim_{y \rightarrow 0} D_{ky}^\alpha E_z^i(x, y), \quad x > 0. \quad (32)$$

The Fourier transform of $f^{1-\alpha}(x)$ is defined as

$$F^{1-\alpha}(q) = \int_{-\infty}^\infty \tilde{f}^{1-\alpha}(\xi) e^{-ikq\xi} d\xi = \int_0^\infty f^{1-\alpha}(x) e^{-ikqx} dx,$$

where $\tilde{f}^{1-\alpha}(\xi) \equiv f^{1-\alpha}(\xi)$ for $\xi > 0$ and $\tilde{f}^{1-\alpha}(\xi) \equiv 0$ for $\xi < 0$.

Then the scattered field will be expressed via the Fourier transform $F^{1-\alpha}(q)$ as

$$E_z^s(x, y) = -i \frac{e^{\pm i\pi\alpha/2}}{4\pi} \int_{-\infty}^\infty F^{1-\alpha}(q) e^{ik(xq + |y|\sqrt{1-q^2})} (1-q^2)^{(\alpha-1)/2} dq. \quad (33)$$

Using the Fourier transform the equation (32) is reduced to the DIE with respect to $F^{1-\alpha}(q)$:

$$\begin{cases} \int_{-\infty}^\infty F^{1-\alpha}(q) e^{ik\xi q} (1-q^2)^{\alpha-1/2} dq = -4\pi e^{i\pi/2(1-\alpha)} \sin^\alpha \theta e^{-ik\xi \cos \theta}, & \xi > 0, \\ \int_{-\infty}^\infty F^{1-\alpha}(q) e^{ik\xi q} dq = 0, & \xi < 0. \end{cases} \quad (34)$$

The kernels in integrals (34) are similar to the ones in DIE (17) obtained for a strip if the constant d_L is equal to 1 ($L = (0, \infty)$ in the case of a half-plane).

For the limit cases of the fractional order $\alpha = 0$ and $\alpha = 1$ these equations are reduced to well known integral equations used for the PEC and PMC half-planes (Veliev, 1999), respectively. In this paper the method to solve DIE (5) is proposed for arbitrary values of $\alpha \in [0, 1]$.

DIE allows an analytical solution in the special case of $\alpha = 0.5$ in the same manner as for a strip with fractional boundary conditions. Indeed, for $\alpha = 0.5$ we obtain the solution for any value of k as

$$F^{0.5}(q) = -2 \sin^{1/2} \theta e^{i\pi/4} \frac{\pi}{k} \delta(q + \cos \theta),$$

$$f^{0.5}(x) = -2 \sin^{1/2} \theta e^{i\pi/4} e^{-ikx \cos \theta}.$$

The scattered field can be found in the following form:

$$E_z^s(x, y) = \frac{i}{2k} e^{\pm i\pi\alpha/2} e^{i\pi/4} \sin^{\alpha-1/2} \theta e^{ik(-\cos \theta x + |y| \sin \theta)}, \quad \alpha = 0.5, \text{ for } y > 0 \text{ (} y < 0 \text{)}.$$

In the general case of $0 < \alpha < 1$ the equations (34) can be reduced to SLAE. To do this we represent the unknown function $\tilde{f}^{1-\alpha}(\xi)$ as a series in terms of the Laguerre polynomials with coefficients f_n^α :

$$\tilde{f}^{1-\alpha}(x) = e^{-x} x^{\alpha-1/2} \sum_{n=0}^{\infty} f_n^\alpha L_n^{\alpha-1/2}(2x). \quad (35)$$

Laguerre polynomials are orthogonal polynomials on the interval $L = (0, \infty)$ with the appropriate weight functions used in (35). It can be shown from (35) that $\tilde{f}^{1-\alpha}(\xi)$ satisfies the following edge condition:

$$\tilde{f}^{1-\alpha}(\xi) = O(\xi^{\alpha-1/2}), \quad \xi \rightarrow 0. \quad (36)$$

For the special cases of $\alpha = 0$ and $\alpha = 1$, the edge conditions are reduced to the well-known equations (Honl et al., 1961) used for a perfectly conducting half-plane.

After substituting (35) into the first equation of (34) we get an integral equation (IE)

$$\sum_{n=0}^{\infty} f_n^\alpha \int_{-\infty}^{\infty} \left[\int_0^{\infty} e^{-t} t^{\alpha-1/2} L_n^{\alpha-1/2}(2t) e^{-ikqt} dt \right] \times e^{ik\xi q} (1-q^2)^{\alpha-1/2} dq = R(\xi), \quad (37)$$

where $R(\xi) = -4\pi e^{i\pi/2(1-\alpha)} \sin^\alpha \theta e^{-ik\xi \cos \theta}$ is known.

Using the representation for Fourier transform of Laguerre polynomials (Prudnikov et al., 1986) we can evaluate the integral over dt as

$$\int_0^{\infty} e^{-t} t^{\alpha-1/2} L_n^{\alpha-1/2}(2t) e^{-ikqt} dt = \int_0^{\infty} e^{-t(1+ikq)} t^{\alpha-1/2} L_n^{\alpha-1/2}(2t) dt = \frac{\Gamma(n+\alpha+1/2)}{\Gamma(n+1)} \frac{(ikq-1)^n}{(ikq+1)^{n+\alpha+1/2}}$$

After some transformations IE (37) is reduced to

$$\sum_{n=0}^{\infty} f_n^\alpha \frac{\Gamma(n+\alpha+1/2)}{\Gamma(n+1)} \int_{-\infty}^{\infty} \frac{(ikq-1)^n}{(ikq+1)^{n+\alpha+1/2}} (1-q^2)^{\alpha-1/2} e^{ik\xi q} dq = R(\xi), \quad \xi > 0. \quad (38)$$

Then we integrate both sides of equation (38) with appropriate weight functions, as

$\int_0^{\infty} (\cdot) e^{-\xi} \xi^{\alpha-1/2} L_m^{\alpha-1/2}(2\xi) d\xi$. Using orthogonality of Laguerre polynomials we get the following SLAE:

$$\sum_{n=0}^{\infty} f_n^\alpha C_{mn}^\alpha = B_m^\alpha, \quad m = 0, 1, 2, \dots, \infty,$$

with matrix coefficients

$$C_{mn}^{\alpha} = \frac{\Gamma(n + \alpha + 1/2)}{\Gamma(n + 1)} \int_{-\infty}^{\infty} \frac{(ikq + 1)^{m-n-\alpha-1/2}}{(ikq - 1)^{n-m-\alpha-1/2}} (1 - q^2)^{\alpha-1/2} dq,$$

$$B_m^{\alpha} = 4\pi e^{-i\pi/2\alpha} \frac{|\sin \theta|^{\alpha} (1 - ik \cos \theta)^m}{(1 + ik \cos \theta)^{m+\alpha+1/2}}.$$

It can be shown that the coefficients f_n^{α} can be found with any desired accuracy by using the truncation of SLAE. Then the function $\tilde{f}^{1-\alpha}(x)$ is found from (35) that allows obtaining the scattered field (33).

4. Diffraction by two parallel strips with fractional boundary conditions

The proposed method to solve diffraction problems on surfaces described by fractional boundary conditions can be applied to more complicated structures. The interest to such structures is related to the resonance properties of scattering if the distance between the strips varies. Two strips of the width $2a$ infinite along the axis z are located in the planes $y = l$ and $y = -l$. Let the E -polarized plane wave $E_z^i(x, y) = e^{-ik(x \cos \theta + y \sin \theta)}$ (1) be the incident field. The total field $E_z = E_z^i + E_z^s$ satisfies fractional boundary conditions on each strip:

$$D_{ky}^{\alpha} E_z(x, y) = 0, \quad y \rightarrow \pm l \pm 0, \quad x \in (-a, a), \quad (39)$$

and Meixner's edge conditions must be satisfied on the edges of both strips ($y = \pm l$, $x \rightarrow \pm a$).

The scattered field $E_z^s(x, y)$ consists of two parts

$$E_z^s(x, y) \equiv E_z^{1s}(x, y) + E_z^{2s}(x, y),$$

where

$$E_z^{js}(x, y_j) \equiv \int_{-a}^a f_j^{1-\alpha}(x') G^{\alpha}(x - x', y_j) dx', \quad j = 1, 2. \quad (40)$$

Here, G^{α} is the fractional Green's function defined in (2). $y_{1,2}$ are the coordinates in the corresponding coordinate systems related to each strip,

$$y_1 = y - l, \quad x_1 = x,$$

$$y_2 = y + l, \quad x_2 = x.$$

Using Fourier transforms, defined as

$$F_j^{1-\alpha}(q) \equiv \int_{-\infty}^{\infty} \tilde{f}_j^{1-\alpha}(\xi) e^{-ikq\xi} d\xi = a \int_{-1}^1 f_j^{1-\alpha}(a\xi) e^{-ikq\xi} d\xi,$$

$$\tilde{f}_j^{1-\alpha}(\xi) \equiv a f_j^{1-\alpha}(a\xi), \quad j = 1, 2,$$

the scattered field is expressed as

$$E_z^{1s}(x, y) = -i \frac{e^{\pm i\pi\alpha/2}}{4\pi} \int_{-\infty}^{\infty} F_1^{1-\alpha}(q) e^{ik[xq + |y-l|\sqrt{1-q^2}]} (1-q^2)^{(\alpha-1)/2} dq, \quad y > l \quad (y < l), \quad (41)$$

$$E_z^{2s}(x, y) = -i \frac{e^{\pm i\pi\alpha/2}}{4\pi} \int_{-\infty}^{\infty} F_2^{1-\alpha}(q) e^{ik[xq + |y+l|\sqrt{1-q^2}]} (1-q^2)^{(\alpha-1)/2} dq, \quad y > -l \quad (y < -l). \quad (42)$$

Fractional boundary conditions (30) correspond to two equations

$$D_{ky}^\alpha E_z(x, y) = 0, \quad y \rightarrow l \pm 0, \quad x \in (-a, a). \quad (43)$$

$$D_{ky}^\alpha E_z(x, y) = 0, \quad y \rightarrow -l \pm 0, \quad x \in (-a, a). \quad (44)$$

After substituting expressions (41) and (42) into the equations (43) and (44) we obtain

$$\begin{aligned} \int_{-\infty}^{\infty} F_1^{1-\alpha}(q) e^{ikxq} (1-q^2)^{\alpha-1/2} dq &= -4\pi i e^{i\pi\alpha/2} \sin^\alpha \theta e^{-ik(x \cos \theta + l \sin \theta)} - \\ &- \int_{-\infty}^{\infty} F_2^{1-\alpha}(q) e^{ik[xq + 2l\sqrt{1-q^2}]} (1-q^2)^{\alpha-1/2} dq \end{aligned} \quad (45)$$

$$\begin{aligned} \int_{-\infty}^{\infty} F_2^{1-\alpha}(q) e^{ikxq} (1-q^2)^{\alpha-1/2} dq &= -4\pi i e^{i\pi/2\alpha} \sin^\alpha \theta e^{-ik(x \cos \theta - l \sin \theta)} - \\ &- \int_{-\infty}^{\infty} F_1^{1-\alpha}(q) e^{ik[xq + 2l\sqrt{1-q^2}]} (1-q^2)^{\alpha-1/2} dq \end{aligned} \quad (46)$$

Multiplying both equations with $e^{-ikx\tau}$ and integrating them in ζ on the interval $[-a, a]$, the system (45), (46) leads to

$$\begin{cases} \int_{-\infty}^{\infty} F_1^{1-\alpha}(q) \frac{\sin ka(q-\tau)}{q-\tau} (1-q^2)^{\alpha-1/2} dq = -4\pi i e^{i\pi\alpha/2} \sin^\alpha \theta \frac{\sin ka(\tau + \cos \theta)}{\tau + \cos \theta} e^{-ikl \sin \theta} - \\ - \int_{-\infty}^{\infty} F_2^{1-\alpha}(q) \frac{\sin ka(q-\tau)}{q-\tau} e^{i2kl\sqrt{1-q^2}} (1-q^2)^{\alpha-1/2} dq \\ \int_{-\infty}^{\infty} F_2^{1-\alpha}(q) \frac{\sin ka(q-\tau)}{q-\tau} (1-q^2)^{\alpha-1/2} dq = -4\pi i e^{i\pi\alpha/2} \sin^\alpha \theta \frac{\sin ka(\tau + \cos \theta)}{\tau + \cos \theta} e^{ikl \sin \theta} - \\ - \int_{-\infty}^{\infty} F_1^{1-\alpha}(q) \frac{\sin ka(q-\tau)}{q-\tau} e^{i2kl\sqrt{1-q^2}} (1-q^2)^{\alpha-1/2} dq \end{cases} \quad (47)$$

Similarly to the method described for the diffraction by one strip, the set (47) can be reduced to a SLAE by presenting the unknown functions $f_j^{1-\alpha}(x)$ as a series in terms of the orthogonal polynomials. We represent the unknown functions $\tilde{f}_j^{1-\alpha}(\xi)$ as series in terms of the Gegenbauer polynomials:

$$\tilde{f}_j^{1-\alpha}(\xi) = (1-\xi^2)^{\alpha-1/2} \sum_{n=0}^{\infty} f_n^{j,\alpha} \frac{1}{\alpha} C_n^\alpha(\xi), \quad j = 1, 2.$$

For the Fourier transforms $F_j^{1-\alpha}(q)$ we have the representations (22). Substituting the representations for $F_j^{1-\alpha}(q)$ into the (47), using the formula (25), then integrating $\int_{-\infty}^{\infty} \frac{J_{m+\alpha}(ka\tau)}{\tau^\alpha} d\tau$ for $m=0,1,2,\dots$, we obtain the following SLAE:

$$\begin{cases} \sum_{n=0}^{\infty} (-i)^n \frac{\Gamma(n+2\alpha)}{\Gamma(n+1)} C_{mn}^{11,\alpha} f_n^{1,\alpha} + \sum_{n=0}^{\infty} (-i)^n \frac{\Gamma(n+2\alpha)}{\Gamma(n+1)} C_{mn}^{12,\alpha} f_n^{2,\alpha} = B_m^{1,\alpha} \\ \sum_{n=0}^{\infty} (-i)^n \frac{\Gamma(n+2\alpha)}{\Gamma(n+1)} C_{mn}^{21,\alpha} f_n^{1,\alpha} + \sum_{n=0}^{\infty} (-i)^n \frac{\Gamma(n+2\alpha)}{\Gamma(n+1)} C_{mn}^{22,\alpha} f_n^{2,\alpha} = B_m^{2,\alpha} \end{cases}, \quad m=0,1,2,\dots$$

where the matrix coefficients are defined as

$$\begin{aligned} C_{mn}^{11,\alpha} &= C_{mn}^{22,\alpha} = \int_{-\infty}^{\infty} \frac{J_{m+\alpha}(ka\tau)}{\tau^\alpha} \frac{J_{n+\alpha}(ka\tau)}{\tau^\alpha} (1-\tau^2)^{\alpha-1/2} d\tau, \\ C_{mn}^{12,\alpha} &= C_{mn}^{21,\alpha} = \int_{-\infty}^{\infty} \frac{J_{m+\alpha}(ka\tau)}{\tau^\alpha} \frac{J_{n+\alpha}(ka\tau)}{\tau^\alpha} e^{i2kl\sqrt{1-\tau^2}} (1-\tau^2)^{\alpha-1/2} d\tau, \\ B_m^{1,\alpha} &= e^{-2ikl\sin\theta} B_m^{2,\alpha} = -2ie^{i\pi\alpha/2} \Gamma(\alpha+1) \sin^\alpha \theta e^{-ikl\sin\theta} (2ka)^\alpha \frac{J_{m+\alpha}(ka\cos\theta)}{(\cos\theta)^\alpha}. \end{aligned}$$

Consider the case of the physical optics approximation, where $ka \gg 1$. In this case we can obtain the solution of (47) in the explicit form. Indeed, using the formula (28) we get

$$\begin{cases} \pi F_1^{1-\alpha}(\tau)(1-\tau^2)^{\alpha-1/2} = \\ = -4\pi i e^{i\pi\alpha/2} \sin^\alpha \theta \frac{\sin ka(\tau + \cos\theta)}{\tau + \cos\theta} e^{-ikl\sin\theta} - \pi F_2^{1-\alpha}(\tau) e^{i2kl\sqrt{1-\tau^2}} (1-\tau^2)^{\alpha-1/2} \\ \pi F_2^{1-\alpha}(\tau)(1-\tau^2)^{\alpha-1/2} = \\ = -4\pi i e^{i\pi\alpha/2} \sin^\alpha \theta \frac{\sin ka(\tau + \cos\theta)}{\tau + \cos\theta} e^{ikl\sin\theta} - \pi F_1^{1-\alpha}(\tau) e^{i2kl\sqrt{1-\tau^2}} (1-\tau^2)^{\alpha-1/2} \end{cases} \quad (48)$$

Finally, we obtain the solution as

$$\begin{cases} F_1^{1-\alpha}(\tau) = 4ie^{i\pi\alpha/2} \sin^\alpha \theta \frac{\sin ka(\tau + \cos\theta)}{\tau + \cos\theta} \frac{1}{(1-\tau^2)^{\alpha-1/2}} \frac{(e^{ikl\sin\theta} e^{i2kl\sqrt{1-\tau^2}} - e^{-ikl\sin\theta})}{(1 - e^{i4kl\sqrt{1-\tau^2}})} \\ F_2^{1-\alpha}(\tau) = 4ie^{i\pi\alpha/2} \sin^\alpha \theta \frac{\sin ka(\tau + \cos\theta)}{\tau + \cos\theta} \frac{1}{(1-\tau^2)^{\alpha-1/2}} \frac{(e^{-ikl\sin\theta} e^{i2kl\sqrt{1-\tau^2}} - e^{ikl\sin\theta})}{(1 - e^{i4kl\sqrt{1-\tau^2}})} \end{cases} \quad (49)$$

Having expressions for $F_j^{1-\alpha}(q)$ we can obtain the physical characteristics. The radiation pattern of the scattered field in the far zone (27) is expressed as

$$\Phi^\alpha(\varphi) = \Phi_1^\alpha(\varphi) + \Phi_2^\alpha(\varphi),$$

where

$$\Phi_1^\alpha(\varphi) = -\frac{i}{4} e^{\pm i\pi/2\alpha} \Gamma_1^{1-\alpha}(\cos\varphi) \sin^\alpha\varphi e^{-ikl\cos\varphi},$$

$$\Phi_2^\alpha(\varphi) = -\frac{i}{4} e^{\pm i\pi/2\alpha} \Gamma_2^{1-\alpha}(\cos\varphi) \sin^\alpha\varphi e^{ikl\cos\varphi}.$$

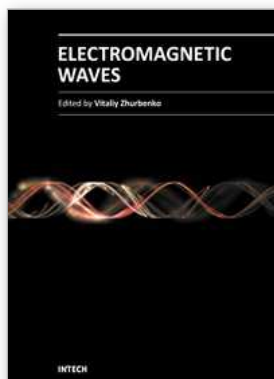
5. Conclusion

The problems of diffraction by flat screens characterized by the fractional boundary conditions have been considered. Fractional boundary conditions involve fractional derivative of tangential field components. The order of fractional derivative is chosen between 0 and 1. Fractional boundary conditions can be treated as intermediate case between well known boundary conditions for the perfect electric conductor (PEC) and perfect magnetic conductor (PMC). A method to solve two-dimensional problems of scattering of the E-polarized plane wave by a strip and a half-plane with fractional boundary conditions has been proposed. The considered problems have been reduced to dual integral equations discretized using orthogonal polynomials. The method allowed obtaining the physical characteristics with a desired accuracy. One important feature of the considered integral equations has been noted: these equations can be solved analytically for one special value of the fractional order equal to 0.5 for any value of frequency. In that case the solution to diffraction problem has an analytical form. The developed method has been also applied to the analysis of a more complicated structure: two parallel strips. Introducing of fractional derivative in boundary conditions and the developed method of solving such diffraction problems can be a promising technique in modeling of scattering properties of complicated surfaces when the order of fractional derivative is defined from physical parameters of a surface.

6. References

- Bateman, H. & Erdelyi, A. (1953). *Higher Transcendental Functions, Volume 2*, McGraw-Hill, New York
- Carlson J.F. & Heins A.E. (1947). The reflection of an electromagnetic plane wave by an infinite set of plates. *Quart. Appl. Math.*, Vol.4, pp. 313-329
- Copson E.T. (1946). On an integral equation arising in the theory of diffraction, *Quart. J. Math.*, Vol.17, pp. 19-34
- Engheta, N. (1996). Use of Fractional Integration to Propose Some 'Fractional' Solutions for the Scalar Helmholtz Equation. A chapter in *Progress in Electromagnetics Research (PIER)*, Monograph Series, Chapter 5, Vol.12, Jin A. Kong, ed. EMW Pub., Cambridge, MA, pp. 107-132
- Engheta, N. (1998). Fractional curl operator in electromagnetic. *Microwave and Optical Technology Letters*, Vol.17, No.2, pp. 86-91
- Engheta, N. (1999). Phase and amplitude of fractional-order intermediate wave, *Microwave and optical technology letters*, Vol.21, No.5

- Engheta, N. (2000). Fractional Paradigm in Electromagnetic Theory, *a chapter in IEEE Press*, chapter 12, pp.523-553
- Hanninen, I.; Lindell, I.V. & Sihvola, A.H. (2006). Realization of Generalized Soft-and-Hard Boundary, *Progress In Electromagnetics Research*, PIER 64, pp. 317-333
- Hilfer, R. (1999). *Applications of Fractional Calculus in Physics*, World Scientific Publishing, ISBN 981-0234-57-0, Singapore
- Honl, H., A.; Maue, W. & Westpfahl, K. (1961). *Theorie der Beugung*, Springer-Verlag, Berlin
- Hope D. J. & Rahmat-Samii Y. (1995). *Impedance boundary conditions in electromagnetic*, Taylor and Francis, Washington, USA
- Lindell I.V. & Sihvola A.H. (2005). Transformation method for Problems Involving Perfect Electromagnetic Conductor (PEMC) Structures. *IEEE Trans. Antennas Propag.*, Vol.53, pp. 3005-3011
- Lindell I.V. & Sihvola A.H. (2005). Realization of the PEMC Boundary. *IEEE Trans. Antennas Propag.*, Vol.53, pp. 3012-3018
- Oldham, K.B. & Spanier, J. (1974). *The Fractional Calculus: Integrations and Differentiations of Arbitrary Order*, Academic Press, New York
- Prudnikov, H.P.; Brychkov, Y.H. & Marichev, O.I. (1986). *Special Functions, Integrals and Series, Volume 2*, Gordon and Breach Science Publishers
- Samko, S.G.; Kilbas, A.A. & Marichev, O.I. (1993), *Fractional Integrals and Derivatives, Theory and Applications*, Gordon and Breach Science Publ., Langhorne
- Senior, T.B.A. (1952). Diffraction by a semi-infinite metallic sheet. *Proc. Roy. Soc. London, Seria A*, 213, pp. 436-458.
- Senior, T.B.A. (1959). Diffraction by an imperfectly conducting half plane at oblique incidence. *Appl. Sci. Res.*, B8, pp. 35-61
- Senior, T.B. & Volakis, J.L. (1995). *Approximate Boundary Conditions in Electromagnetics*, IEE, London
- Uflyand, Y.S. (1977). The method of dual equations in problems of mathematical physics [in russian]. *Nauka*, Leningrad
- Veliev, E.I. & Shestopalov, V.P. (1988). A general method of solving dual integral equations. *Sov. Physics Dokl.*, Vol.33, No.6, pp. 411-413
- Veliev, E.I. & Veremey, V.V. (1993). Numerical-analytical approach for the solution to the wave scattering by polygonal cylinders and flat strip structures. *Analytical and Numerical Methods in Electromagnetic Wave Theory*, M. Hashimoto, M. Idemen, and O. A. Tretyakov (eds.), Chap. 10, Science House, Tokyo
- Veliev, E.I. (1999). Plane wave diffraction by a half-plane: a new analytical approach. *Journal of electromagnetic waves and applications*, Vol.13, No.10, pp. 1439-1453
- Veliev, E.I. & Engheta, N. (2003). Generalization of Green's Theorem with Fractional Differintegration, *IEEE AP-S International Symposium & USNC/URSI National Radio Science Meeting*
- Veliev, E.I.; Ivakhnychenko, M.V. & Ahmedov, T.M. (2008). Fractional boundary conditions in plane waves diffraction on a strip. *Progress In Electromagnetics Research*, Vol.79, pp. 443-462
- Veliev, E.I.; Ivakhnychenko, M.V. & Ahmedov, T.M. (2008). Scattering properties of the strip with fractional boundary conditions and comparison with the impedance strip. *Progress In Electromagnetics Research C*, Vol.2, pp. 189-205



Electromagnetic Waves

Edited by Prof. Vitaliy Zhurbenko

ISBN 978-953-307-304-0

Hard cover, 510 pages

Publisher InTech

Published online 21, June, 2011

Published in print edition June, 2011

This book is dedicated to various aspects of electromagnetic wave theory and its applications in science and technology. The covered topics include the fundamental physics of electromagnetic waves, theory of electromagnetic wave propagation and scattering, methods of computational analysis, material characterization, electromagnetic properties of plasma, analysis and applications of periodic structures and waveguide components, and finally, the biological effects and medical applications of electromagnetic fields.

How to reference

In order to correctly reference this scholarly work, feel free to copy and paste the following:

Eldar Veliev, Turab Ahmedov, Maksym Ivakhnychenko (2011). Fractional Operators Approach and Fractional Boundary Conditions, *Electromagnetic Waves*, Prof. Vitaliy Zhurbenko (Ed.), ISBN: 978-953-307-304-0, InTech, Available from: <http://www.intechopen.com/books/electromagnetic-waves/fractional-operators-approach-and-fractional-boundary-conditions>

INTeCH
open science | open minds

InTech Europe

University Campus STeP Ri
Slavka Krautzeka 83/A
51000 Rijeka, Croatia
Phone: +385 (51) 770 447
Fax: +385 (51) 686 166
www.intechopen.com

InTech China

Unit 405, Office Block, Hotel Equatorial Shanghai
No.65, Yan An Road (West), Shanghai, 200040, China
中国上海市延安西路65号上海国际贵都大饭店办公楼405单元
Phone: +86-21-62489820
Fax: +86-21-62489821