

# Essential Electromagnetism

Raymond John Protheroe



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1<sup>st</sup> edition

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## Preface

“Essential Electromagnetism” and “Essential Electrodynamics” (also to be published by Ventus) are intended to be resources for students taking electromagnetism courses while pursuing undergraduate studies in physics and engineering. Due to limited space available in this series, it is not possible to go into the material in great depth, so I have attempted to encapsulate what I consider to be the essentials. This book does not aim to replace existing textbooks on these topics of which there are many excellent examples, several of which are listed in the bibliography. Nevertheless, if appropriately supplemented, this book and “Essential Electrodynamics” could together serve as a textbook for 2nd and 3rd year electromagnetism courses at Australian and British universities, or for junior/senior level electromagnetism courses at American universities/colleges.

The book assumes a working knowledge of partial differential equations, vectors and vector calculus as would normally be acquired in mathematics courses taken by physics and engineering students. However, very brief introductions to vectors, vector calculus and index notation are given in the appendices. Some of the mathematical derivations have been relegated to the appendices, and some of those are carried out using index notation which is briefly introduced in Appendix D, but elsewhere in the book manipulation of equations involving vector differential calculus is done using standard vector calculus identities (also given in the appendices).

The presentation of the subject material in this book is conventional, starting with forces between charges, electric field, Coulomb’s law, electric flux, Gauss’ law and the electrostatic potential. Chapter 2 is on Poisson’s and Laplace’s equations, and their solution. The method of images is applied to simple examples in plane, cylindrical and spherical geometry. Laplace’s equation is solved analytically in Cartesian coordinates for the cases where the boundaries are orthogonal planes, and in spherical coordinates where the boundary surface is a sphere; these being the most commonly-encountered problems involving Laplace’s equation. Solution of Laplace’s equation in cylindrical coordinates is not included, but may be found in most applied mathematics texts. A numerical finite-difference method is described for solving Laplace’s equation. This method is applicable whether or not the boundary conditions allow analytic solutions.

Chapter 3 is on the multipole expansion of the electrostatic potential, up to and including the quadrupole term. Chapter 4 is on macroscopic and microscopic dielectric theory, starting with the polarisation field, polarisation charges, Gauss’ law and the displacement field, susceptibility and permittivity, and the boundary conditions on the electric and displacement fields. Orien-

tational polarisability of molecules, electronic polarisability of non-polar molecules and ionic polarisability of crystals are discussed. Finally, the Clausius-Mossotti formula which connects the microscopic and macroscopic theories is derived.

Chapter 5 deals with the magnetic field, its causes, magnetic forces and the magnetic flux. The magnetic vector potential is derived from the Biot-Savart law, and a derivation of Ampère's law follows, together with examples of its use in solving problems in magnetostatics. Finally the multipole expansion of the vector potential up to the dipole term is performed, and magnetic dipoles are discussed together with their magnetic field and the torques and forces they experience in a magnetic field.

Chapter 6 focuses on the magnetism of materials, starting with the magnetisation field, magnetisation currents and their inclusion in Ampère's law, and the introduction of the auxiliary field  $\mathbf{H}$ . Next, susceptibility and permeability are defined, and the boundary conditions on the magnetic and auxiliary fields are derived. Finally, the causes of magnetisation are discussed, namely orientational polarisability of atomic magnetic dipoles associated with unpaired electron spins causing paramagnetism, and the quantum mechanical exchange interaction in some materials giving rise to unpaired electron spins lining up in the same direction as in neighbouring atoms, and leading to ferromagnetism.



Each chapter is followed by several exercise problems, and solutions to these problems are to be published separately by Ventus as “Essential Electromagnetism - Solutions”. I suggest you attempt these exercises before looking at the solutions.

I hope you find this book useful. If you find typos or errors I would appreciate you letting me know so that I can fix them in the next edition. Suggestions for improvement are also welcome – please email them to me at [protheroe.essentialphysics@gmail.com](mailto:protheroe.essentialphysics@gmail.com).

This book was mainly written in the evenings and I would like to thank my family for their support and forbearance. It is dedicated to the memory of my parents, who stimulated my interest in science.

Raymond John Protheroe,  
School of Chemistry and Physics, The University of Adelaide, Australia  
December 2012

# 1 Electrostatics

## Learning objectives

- To learn the concepts of electric charge, charge density and the electric field, and how they are related through Coulomb's law and the principle of superposition as applied to forces.
- To understand the concepts of electric field lines, and electric flux as the surface integral of the electric field.
- To be able to derive Gauss' law in integral and differential form, and be able to use it to obtain the electric field from symmetrical charge distributions.
- To know why some materials are conductors and others are insulators (dielectrics), why in electrostatics the electric field is zero inside a conductor, that any charge present resides on the surface of a conductor, and that the electric field just outside a conductor is normal to the surface.
- To understand what the electrostatic potential is, how it may be obtained from the electric field, and that the electric field is the negative gradient of the potential.
- To be able to find the work done to assemble a charge distribution, to understand that the potential energy is stored in the electric field, and to be able to find its energy density.

### 1.1 Electric charge, field and flux

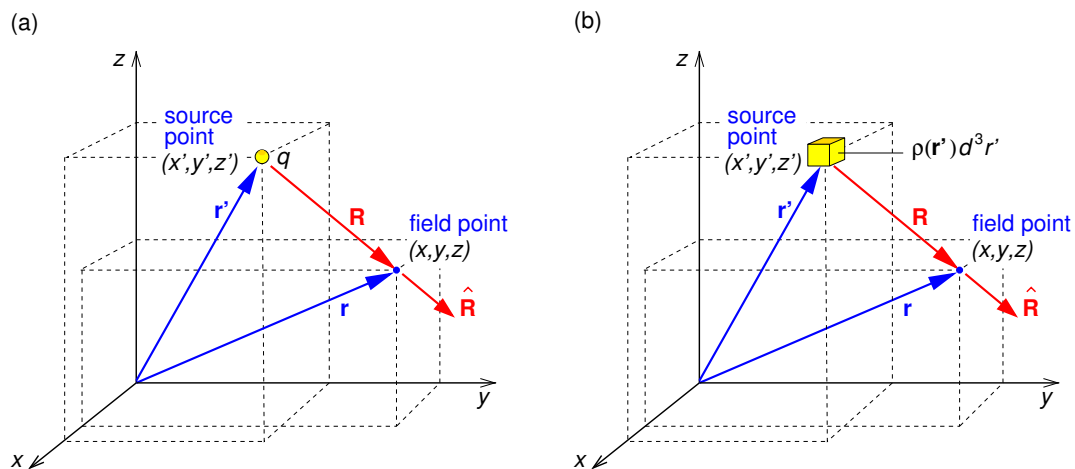
In 1785 French physicist Charles-Augustin de Coulomb (1736–1806) published his law, which may be stated as: the force exerted on point charge  $Q$  by point charge  $q$  a distance  $R$  away has magnitude proportional to  $Qq/R^2$  and is directed along the line between the two charges. American statesman and scientist Benjamin Franklin (1706–1790) was responsible for naming the charge we now associate with electrons “negative” and the charge we now associate with protons “positive”, and was the first to state the law of conservation of charge. Electric charge is measured in coulombs (C).

We may consider the force on charge  $Q$  as being due to the *electric field*  $\mathbf{E}$  of charge  $q$  acting on charge  $Q$ , i.e.  $\mathbf{F} = Q\mathbf{E}$ . The electric field is defined as the *force per unit test charge* (unit:

$\text{N C}^{-1}$  or  $\text{V m}^{-1}$ ); a test charge being an infinitesimal point charge placed where the electric field is to be evaluated. From Coulomb's law, the electric field of a point charge is

$$\mathbf{E}(\mathbf{r}) = \frac{q\hat{\mathbf{R}}}{4\pi\epsilon_0 R^2} \quad (1.1)$$

where  $\mathbf{R} = (\mathbf{r} - \mathbf{r}')$ ,  $\mathbf{r}'$  is the *source point* and  $\mathbf{r}$  is the *field point* as in Fig. 1.1(a), and  $\epsilon_0 = 8.85418782 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$  is the permittivity of free space; more usually its units are given as  $(\text{F m}^{-1})$  where F stands for the farad which is the unit of capacitance, and is named after English chemist and physicist Michael Faraday (1791–1867).

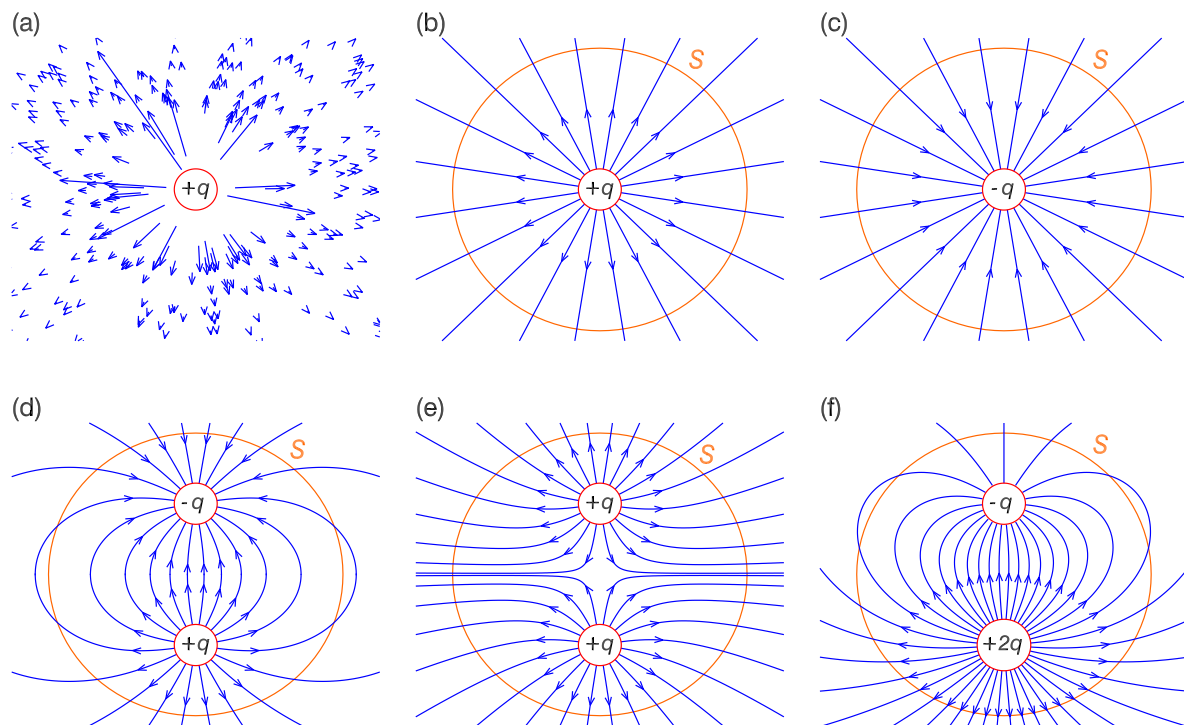


**Figure 1.1:** Geometry for discussing (a) Coulomb's law for point charge  $q$ , and (b) Coulomb's law for charge density  $\rho(\mathbf{r})$ .

Since the electric field is the force per unit test charge, it obeys the principle of superposition just like forces do. Hence, for a continuous charge distribution with *volume charge density*  $\rho(\mathbf{r})$  ( $\text{C m}^{-3}$ ) the electric field is (Fig. 1.1b)

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')\hat{\mathbf{R}}}{R^2} d^3r'. \quad (1.2)$$

We may think of the electric field of, for example, an isolated positive point charge (Fig. 1.2a), in terms of *field lines* which emerge isotropically from positive charges (Fig. 1.2b). The number of field lines per unit area crossing a concentric sphere of radius  $r$  is then  $n/(4\pi r^2) \propto 1/r^2$  where  $n \propto q$  is the number of field lines originating at  $q$ . At any location, the electric field's direction is tangential to the field line at that location.



**Figure 1.2:** (a) Electric field vectors due to a positive charge, where the length of an arrow is proportional to  $E$  at the location of its tail. (b) Electric field lines starting at an isolated positive charge. (c) Electric field lines ending on an isolated negative charge. (d) Electric field lines starting on positive charge  $+q$  and ending on a nearby negative charge  $-q$ . (e) Electric field lines starting on two nearby positive charges  $+q$  repel each other. (f) Field lines for the case of two opposite charges of unequal magnitude. “ $S$ ” represents a closed surface surrounding all charges present. Note that in these, and subsequent, figures the point charges are located at the centres of the white discs on which their charge is written. (The field line tracing for all the images in this book which show electric or magnetic field field lines has been done using a numerical scheme based on the 2nd order Runge-Kutta method for initial value problems described in Appendix F.)

### 1.1.1 Electric flux

The electric flux through a surface  $S$  is defined by

$$\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{S} = \int_S \mathbf{E} \cdot \hat{\mathbf{n}} dS \quad (1.3)$$

and this is proportional to the net number of field lines passing out through  $S$ , i.e. the number going out minus the number going in (Figs. 1.2b–c). In the flux integral the vector surface element  $d\mathbf{S}$  is the the scalar surface element  $dS$  multiplied by the normal unit vector  $\hat{\mathbf{n}}$  at the

surface, i.e.  $d\mathbf{S} \equiv \hat{\mathbf{n}} dS$ . In the case of a closed surface the direction of  $d\mathbf{S}$  is always defined to be outward from the volume, but otherwise there is an ambiguity and the direction must be specified.

For a spherical surface of radius  $r$  centred on the origin as illustrated in Fig. 1.3(a)

$$dS = (r d\theta)(r \sin \theta d\phi) = r^2 \sin \theta d\theta d\phi = r^2 |d(\cos \theta)| d\phi \quad (1.4)$$

and  $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ . Thus, for charge  $+q$  at the origin, the flux outward through the sphere of radius  $r$  centred on the origin is

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \int_0^{2\pi} \int_{-1}^1 \left( \frac{q \hat{\mathbf{r}}}{4\pi\epsilon_0 r^2} \right) \cdot \hat{\mathbf{r}} r^2 d(\cos \theta) d\phi = \frac{q}{\epsilon_0} \quad (1.5)$$

Of course, we could obtain this result more easily by exploiting the spherical symmetry of the problem — the surface integral is simply the product of  $E = q/(4\pi\epsilon_0 r^2)$  with the total area of the sphere  $S = 4\pi r^2$ .

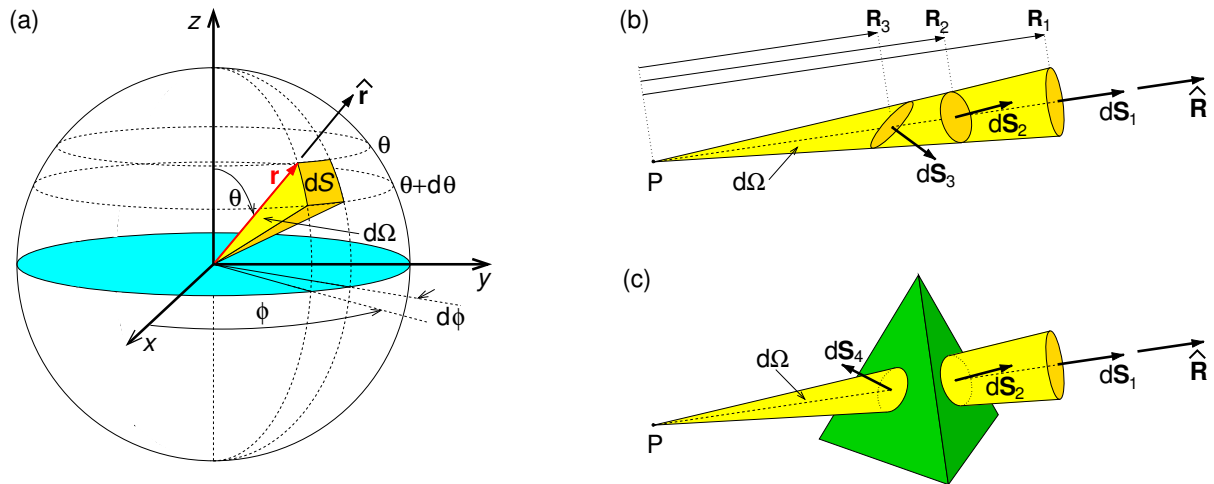
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**Figure 1.3:** (a) Surface element  $dS$  of a sphere of radius  $r$  centred on the origin, illustrating spherical coordinates  $(r, \theta, \phi)$ . (b) Surfaces  $dS_1$ ,  $dS_2$  and  $dS_3$  subtend the same solid angle  $d\Omega$  at point P. (c) Surfaces  $dS_2$  and  $dS_4$  are parts of the closed surface of a tetrahedron, with  $dS_2$  subtending solid angle  $d\Omega$  at point P, whereas  $dS_4$  subtends solid angle  $-d\Omega$  at point P.

Another way of considering the electric flux through a spherical surface is in terms of the *solid angle* subtended by surface element  $dS$  at the centre of the sphere. The solid angle can be defined in an analogous way to an ordinary angle. An arc of length  $\ell$  which is part of a circle of radius  $r$  subtends angle  $\theta = \ell/r$  radians (rad) at the centre, so that if the arc were closed (i.e. a circle) the angle would be  $\theta = 2\pi r/r = 2\pi$  rad. In a similar way area  $dS$  which is part of a spherical surface of radius  $r$  subtends solid angle  $d\Omega = dS/r^2$  steradians (sr) at the centre, so that if the surface were closed (i.e. a sphere) the solid angle would be  $\Omega = 4\pi r^2/r^2 = 4\pi$  sr. Hence the solid angle subtended at the centre by surface element  $dS$  in Fig. 1.3(a) is

$$d\Omega = \frac{dS}{r^2} = \sin \theta \, d\theta \, d\phi = |d(\cos \theta)| \, d\phi. \quad (1.6)$$

In Fig. 1.3(b) the solid angles subtended at point P by small surfaces  $dS_1$ ,  $dS_2$  and  $dS_3$  are obviously the same – it is the solid angle  $d\Omega$  within the cone which has its apex at P. Notice that only  $dS_1$  could be part of a spherical surface centred on P, in this case having radius  $R_1$ , whereas  $dS_2$  and  $dS_3$  are not centred on P. However, what counts is the projected area, so that

$$d\Omega_1 = \frac{d\mathbf{S}_1 \cdot \hat{\mathbf{R}}_1}{R_1^2}, \quad d\Omega_2 = \frac{d\mathbf{S}_2 \cdot \hat{\mathbf{R}}_2}{R_2^2}, \quad d\Omega_3 = \frac{d\mathbf{S}_3 \cdot \hat{\mathbf{R}}_3}{R_3^2}, \quad (1.7)$$

where  $\mathbf{R}_1$ ,  $\mathbf{R}_2$  and  $\mathbf{R}_3$  are separation vectors from P to  $dS_1$ ,  $dS_2$  and  $dS_3$ . For  $dS_1$ ,  $dS_2$  and  $dS_3$  in Fig. 1.3(b)  $d\Omega_1 = d\Omega_2 = d\Omega_3 = d\Omega$ .



Supposing  $dS_1$ ,  $dS_2$  and  $dS_3$  are parts of closed surfaces  $S_1$ ,  $S_2$  and  $S_3$  of arbitrary shape surrounding  $P$ , then the solid angles these surfaces subtend at  $P$  are

$$\oint_{S_1} \frac{d\mathbf{S}_1 \cdot \hat{\mathbf{R}}_1}{R_1^2} = \oint_{S_2} \frac{d\mathbf{S}_2 \cdot \hat{\mathbf{R}}_2}{R_2^2} = \oint_{S_3} \frac{d\mathbf{S}_3 \cdot \hat{\mathbf{R}}_3}{R_3^2} = \oint d\Omega = 4\pi \quad (1.8)$$

On the other hand, if point  $P$  isn't inside the closed surface (e.g.  $P$  is outside the tetrahedron in Fig. 1.3c) then  $\oint_S \frac{d\mathbf{S} \cdot \hat{\mathbf{R}}}{R^2} = 0$ ; notice that the contributions from  $d\mathbf{S}_2$  and  $d\mathbf{S}_4$  cancel each other out as  $d\Omega_4 = -d\Omega_2$ .

Returning to Fig. 1.3(b), the electric fluxes due to charge  $q$  at  $P$  out through surrounding closed surfaces  $S_1$ ,  $S_2$  and  $S_3$  (of which  $dS_1$ ,  $dS_2$  and  $dS_3$  are surface elements) are

$$\frac{q}{4\pi\epsilon_0} \oint_{S_1} \frac{d\mathbf{S}_1 \cdot \hat{\mathbf{R}}}{R_1^2} = \frac{q}{4\pi\epsilon_0} \oint_{S_2} \frac{d\mathbf{S}_2 \cdot \hat{\mathbf{R}}}{R_2^2} = \frac{q}{4\pi\epsilon_0} \oint_{S_3} \frac{d\mathbf{S}_3 \cdot \hat{\mathbf{R}}}{R_3^2} = \frac{q}{4\pi\epsilon_0} \oint d\Omega = \frac{q}{\epsilon_0}, \quad (1.9)$$

and this is the same for *all* closed surfaces surrounding charge  $q$ .

We can also think of this in terms of the number of field lines leaving a charge per unit solid angle being isotropic (the same in all directions), and the electric flux through a surface being proportional to the number of field lines which in turn is proportional to the charge. Then the electric flux out through a closed surface is proportional to the solid angle it subtends, which is  $4\pi$  if the charge is inside the closed surface, or zero if the charge is outside. If we double the charge the flux would double, or if we add additional positive charges inside the closed surface the electric flux would always equal the sum of the charges inside, divided by  $\epsilon_0$ .

Since the direction along a field line is identical to that of the field direction, field lines must start on positive charges. Similarly, they must end on negative charges (or extend to infinity). Examples of field lines for different charge configurations were already shown in Figs. 1.2(d)–(f). Consider closed surface  $S$  surrounding all of the charge in Figs. 1.2(d)–(f). For Fig. 1.2(d) the net number of field lines *leaving*  $S$  is zero as is the *net* charge inside, so  $\Phi_E = (q - q)/\epsilon_0 = 0$ . For Fig. 1.2(e)  $\Phi_E = (q + q)/\epsilon_0 = 2q/\epsilon_0$ , while for Fig. 1.2(f) for which twice as many field lines leave surface  $S$  as enter it  $\Phi_E = (2q - q)/\epsilon_0 = q/\epsilon_0$ .

### 1.1.2 Gauss' law

This law, named after German mathematician and physicist Johann Carl Friedrich Gauss (1777–1855), is a formalisation of the result of the previous section. *Gauss' law* in integral form is

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} Q_{\text{encl}} \quad (1.10)$$

where  $Q_{\text{encl}}$  is the net charge enclosed by surface  $S$ . Applying Gauss' theorem (Eq. C.19) we get

$$\int_V \nabla \cdot \mathbf{E} d^3r = \int_V \frac{1}{\varepsilon_0} \rho d^3r \quad (1.11)$$

where surface  $S$  encloses volume  $V$ , and  $\rho(\mathbf{r})$  is the charge density ( $\text{C m}^{-3}$ ). Equating integrands we obtain Gauss Law in differential form

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}. \quad (1.12)$$



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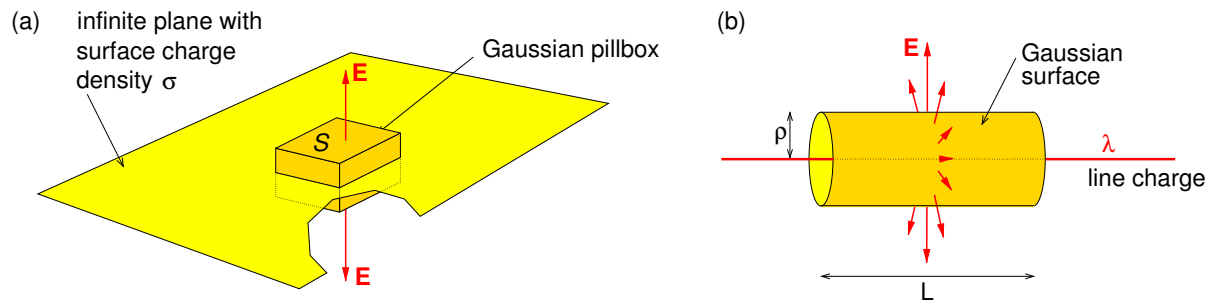
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Where the symmetry permits, Gauss' law in integral form (Eq. 1.10) gives a powerful method of finding the electric field. The trick is to find a volume satisfying: (i) surfaces are either normal to or parallel to the electric field, (ii) where the electric field is normal it is constant over that surface, (iii) the charge inside can be easily calculated. In the case of surface charges it is useful to construct a *Gaussian pillbox* with narrow sides and top and bottom surfaces parallel to the surface containing the charge. Classic examples include the infinite plane with uniform surface charge density  $\sigma$  ( $\text{C m}^{-2}$ ) shown in Fig. 1.4(a), and the line charge density  $\lambda$  ( $\text{C m}^{-1}$ ) of infinite length shown in Fig. 1.4(b).



**Figure 1.4:** (a) Infinite plane with surface charge density  $\sigma$ , showing a Gaussian pillbox straddling the plane and whose upper and lower surfaces are parallel to the plane. (b) Section of infinite line charge surrounded by a Gaussian cylindrical surface.

For the case of the infinite plane, from symmetry arguments the direction of the electric field must be normal to the plane, and for positive charge,  $\mathbf{E}$  must be pointing away from the plane. Then, for area  $S$  of the infinite plane the charge inside the pill-box is  $\sigma S$  so that

$$|\mathbf{E}|2S = \frac{\sigma S}{\epsilon_0}. \quad \therefore \quad \mathbf{E} = \frac{\sigma}{2\epsilon_0} \hat{\mathbf{n}}, \quad (1.13)$$

where the  $2S$  comes from the pillbox having both an upper and a lower surface.

For an infinite line charge  $\lambda$  we can use Gauss' law for a cylindrical Gaussian surface of cylindrical coordinate radius  $\rho$  (Fig. 1.4b). Length  $L$  of the cylinder contains charge  $\lambda L$  so that

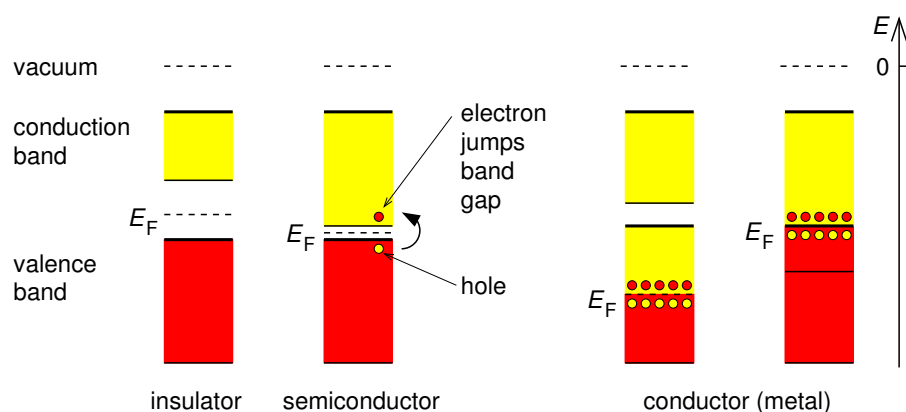
$$|\mathbf{E}|2\pi\rho L = \frac{\lambda L}{\epsilon_0}. \quad \therefore \quad \mathbf{E} = \frac{\lambda}{2\pi\rho\epsilon_0} \hat{\boldsymbol{\rho}}. \quad (1.14)$$

### 1.1.3 Conductors and insulators

To understand why some materials are conductors and some are insulators, requires a brief excursion into quantum mechanics. In a solid, it is the outermost electrons in each atom, the

valence electrons, which are potentially free to move in the quantum well of the volume of the solid which has embedded in it a 3D periodic potential due to the ions in the crystal lattice. As a result, the discrete energy levels of the atoms in the solid merge into separate bands of closely spaced energy levels, possibly with a substantial energy gap between bands.

The energy levels are populated with all the electrons in the solid starting with the lowest levels, but with a maximum of two electrons per level (one with spin up and the other spin down) as allowed by the Pauli exclusion principle. As one level is filled, remaining electrons go in the next level, and so on. The energy of the highest filled level is called the Fermi energy,  $E_F$ . Of course, at finite temperature there is a spread of electron energies above and below the Fermi energy, except when the band is completely filled and the energy band gap is too large. Where  $E_F$  is within a band determines whether the material is a conductor, semiconductor or insulator (Fig. 1.5).



**Figure 1.5:** Electronic band structure of insulators, semiconductors and conductors. Red represents filled levels, and yellow represents vacant levels.

Materials with only one valence electron per atom typically half fill the valence band, whereas materials with two valence electrons per atom fill the valence band. If the Fermi energy is in the middle of the band then lots of electrons with energies near the Fermi energy are able to move to nearby vacant levels in response to an electric field, as this takes only a tiny amount of energy, and are therefore able to move freely throughout the solid, making the material a conductor. Also, if there are exactly enough electrons to fill the valence band but the conduction band overlaps it, then electrons can easily move to the conduction band and the material is also a conductor.

In contrast if the Fermi energy is at the top of a band, i.e. the band is full and there are no electrons in the next band, the material is an insulator. In insulators electrons are trapped in their energy levels; it takes a lot of energy (the band gap energy) to jump to the next level.

In semiconductors the band gap is sufficiently small that a few electrons are able to jump the band gap as a result of thermal excitation and/or in response to an electric field. In the process they leave behind vacant levels in the valence band or “holes” which act like positive electrons and contribute to conduction, just as the electrons that jumped to the conduction band, thereby causing the material to become semiconducting. Semiconductors used to make electronic components are typically “doped” with impurity ions which introduce energy levels within the band gap.

In a conductor, electrons move in response to an electric field, which in electrostatics is due to other charges, and will keep moving until they reach the surface where they will redistribute themselves until the electric field inside the conductor vanishes. Thus, in electrostatics the entire conductor has the same potential, and its surface is an equipotential surface.

Early experiments involved rubbing glass with silk, or rubbing amber with fur. What is going on here is that ruptured chemical bonds on the surface of one material leave a charge imbalance, and when that material is in contact with a different material with a different charge imbalance electrons will flow from one material to the other. It is the contact area that is important in charge transfer, rubbing just increases the contact area. The consequent displacement of

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electrons from the glass to the silk leaves the glass positively charged, and from the fur to the amber leaves the amber negatively charged. Touching the charged glass to an isolated conductor would cause some electrons to flow from the conductor to the glass leaving the conductor positively charged, whereas touching the conductor with charged amber would cause some electrons to flow from the amber to the conductor leaving it negatively charged.

## 1.2 The electrostatic potential

The work done,  $W$ , in moving a particle of charge  $Q$  from point A to point B along an arbitrary path in the electric field of charge  $q$  located at  $\mathbf{r}_q$  is

$$W = - \int_A^B Q \mathbf{E}(\mathbf{r}) \cdot d\mathbf{r} = - \int_A^B Q \frac{q \hat{\mathbf{R}}}{4\pi\epsilon_0 R^2} \cdot d\mathbf{r} = - \int_A^B Q \left[ -\nabla \left( \frac{q}{4\pi\epsilon_0 R} \right) \right] \cdot d\mathbf{r} \quad (1.15)$$

where  $\mathbf{R} = (\mathbf{r} - \mathbf{r}_q)$ , and we have used an identity for  $\nabla R^{-1}$  (see Appendix B.1). Because  $(\nabla R^{-1}) \cdot d\mathbf{r}$  is an exact differential,  $W$  is independent of the path taken from A to B. Applying the principle of superposition for electric fields, we see that this would be true for any distribution of charges, and so for any electrostatic field.

We have just shown that  $\mathbf{E}$  is conservative in electrostatics. A conservative field is one for which its curl is zero, or equivalently it's line-integral around a closed loop is zero. Thus, we may define an electrostatic *potential difference*

$$V_B - V_A = \frac{W}{Q} = - \int_A^B \mathbf{E}(\mathbf{r}) \cdot d\mathbf{r}. \quad (1.16)$$

In general, we may define the *electrostatic potential*

$$V(\mathbf{r}) = - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}' \quad (1.17)$$

such that

$$\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r}). \quad (1.18)$$

The choice of the *reference point*  $\mathbf{r}_0$ , where the potential is *defined* to be zero, is arbitrary and does not effect the value of  $\mathbf{E}(\mathbf{r})$ . It is often chosen to be “at infinity”. If we set  $r_0 = \infty$  we get



the potential for point charge  $q$  at the origin, and for point charge  $q'$  at location  $\mathbf{r}'$

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r}, \quad \text{and} \quad V(\mathbf{r}) = \frac{q'}{4\pi\epsilon_0 R}, \quad (1.19)$$

respectively, where, as usual,  $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ . Applying the principle of superposition, the potentials due to volume and surface charge distributions  $\rho(\mathbf{r})$  and  $\sigma(\mathbf{r})$  are

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{R} d^3r', \quad V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint \frac{\sigma(\mathbf{r}')}{R} dS'. \quad (1.20)$$

Imagine plotting equipotential surfaces – if  $V$  only depends on say  $x$  and  $y$  you could plot a contour map and think of high values of  $V$  corresponding to hills, and low values as valleys. Then  $\mathbf{E}$  at a particular location points down the steepest gradient, and is perpendicular to the contour lines. This also means that the electric field (and field lines) are normal to equipotential surfaces.

### 1.2.1 Energy stored in the electric field

Consider charge,  $q_1$  located at  $\mathbf{r}_1$ . The work done to move a second charge  $q_2$  from infinity to  $\mathbf{r}_2$ , where the potential (due to the other charge) is  $V_2 \equiv V(\mathbf{r}_2)$ , is

$$W = q_2 V_2 = q_2 \frac{q_1}{4\pi\epsilon_0 |\mathbf{r}_2 - \mathbf{r}_1|} = q_1 \frac{q_2}{4\pi\epsilon_0 |\mathbf{r}_1 - \mathbf{r}_2|} = q_1 V_1 = \frac{1}{2} [q_2 V_2 + q_1 V_1]. \quad (1.21)$$

In general the work done to bring together a group of  $N$  charges is

$$W = \frac{1}{2} \sum_{i=1}^N q_i V_i \quad (1.22)$$

where  $V_i$  is the potential due to all charges *except* charge  $q_i$ . By analogy, the work done to form a *continuous* charge distribution is

$$W = \frac{1}{2} \int_{\text{all space}} \rho(\mathbf{r}) V(\mathbf{r}) d^3r. \quad (1.23)$$

This formula cannot be used for point charges as the potential would be infinite at the locations of the charges. In nature this is not a problem as point charges (e.g. electrons) become fuzzy due the wave nature of particles (Heisenberg uncertainly principle).

The energy is stored in the electric field, as we can find by rewriting Eq. 1.23 as

$$W = \frac{1}{2} \int_{\text{all space}} \epsilon_0 \nabla \cdot \mathbf{E}(\mathbf{r}) V(\mathbf{r}) d^3r, \quad (1.24)$$

$$= \frac{1}{2} \int_{\text{all space}} \epsilon_0 \nabla \cdot [\mathbf{E}(\mathbf{r}) V(\mathbf{r})] d^3r - \frac{1}{2} \int_{\text{all space}} \epsilon_0 \mathbf{E}(\mathbf{r}) \cdot \nabla V(\mathbf{r}) d^3r, \quad (1.25)$$

$$= \frac{1}{2} \oint_{S \text{ at } \infty} \epsilon_0 [\mathbf{E}(\mathbf{r}) V(\mathbf{r})] \cdot d\mathbf{S} + \frac{1}{2} \int_{\text{all space}} \epsilon_0 \mathbf{E}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) d^3r, \quad (1.26)$$

$$= \int_{\text{all space}} \frac{\epsilon_0}{2} \mathbf{E}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) d^3r, \quad (1.27)$$

where we have used a product rule (Eq. I.8) to integrate by parts, and Gauss' theorem (Eq. C.19) to replace the volume integral of  $\nabla \cdot (\mathbf{E}V)$  over all space by the surface integral of  $\mathbf{E}V$  over the closed surface at infinity — this is zero as  $EV \sim r^{-3}$  while  $S \sim r^2$  as  $r \rightarrow \infty$ . Hence, we see



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that the energy density stored in an electric field is

$$u_E(\mathbf{r}) = \frac{\varepsilon_0}{2} \mathbf{E}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}) = \frac{\varepsilon_0}{2} [E(\mathbf{r})]^2. \quad (1.28)$$

## Summary of Chapter 1

### Coulomb's law

- Force on charge  $Q$  by charge  $q$  a distance  $R$  away is proportional to  $Qq/R^2$ .
- Force on  $Q$  is due to the *electric field*  $\mathbf{E}$  of  $q$  acting on  $Q$ , i.e.  $\mathbf{F} = Q\mathbf{E}(\mathbf{r}_Q)$ .
- Electric field is defined as the *force per unit test charge* (unit:  $\text{V m}^{-1}$ ).
- Electric field obeys the superposition principle.
- For a point source  $q$  (C) or *volume charge density*  $\rho(\mathbf{r}')$  ( $\text{C m}^{-3}$ ) at the *source point*  $\mathbf{r}'$  the electric field at the *field point*  $\mathbf{r}$  is

$$\mathbf{E}(\mathbf{r}) = \frac{q\hat{\mathbf{R}}}{4\pi\varepsilon_0 R^2}, \quad \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}')\hat{\mathbf{R}}}{R^2} d^3r'. \quad (1.29)$$

where  $\mathbf{R} = (\mathbf{r} - \mathbf{r}')$ .

- $\varepsilon_0 = 8.85418782 \times 10^{-12} \text{ F m}^{-1}$  is the permittivity of free space.

### Electric flux and Gauss' law

- Electric flux through surface  $S$  is defined by  $\Phi_E = \int_S \mathbf{E} \cdot d\mathbf{S}$ .
- Gauss' law:  $\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} Q_{\text{encl}}, \quad \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}.$

### Electrostatic potential

- $\mathbf{E}$  is conservative in electrostatics and we may define an *electrostatic potential*

$$V(\mathbf{r}) = - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}', \quad \mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r}). \quad (1.30)$$

- The choice of *reference point*  $\mathbf{r}_0$ , where the potential is *defined* to be zero, is arbitrary and does not effect  $\mathbf{E}$ . It is often chosen to be “at infinity”

### Energy stored in the electric field

- The work done to bring together  $N$  charges, or a continuous charge distribution  $\rho(\mathbf{r})$  is

$$W = \frac{1}{2} \sum_{i=1}^N q_i V(\mathbf{r}_i), \quad W = \frac{1}{2} \int_{\text{all space}} \rho(\mathbf{r}) V(\mathbf{r}) d^3r. \quad (1.31)$$

- The energy density stored in an electric field is  $u_E(\mathbf{r}) = \frac{\epsilon_0}{2} \mathbf{E}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})$ .

### Exercises on Chapter 1

- 1–1 The surface of a non-conducting sphere of radius  $a$  centred on the origin has surface charge density  $\sigma(a, \theta, \phi) = \sigma_0 \cos \theta$  and is uniformly filled with charge of density  $\rho_0$ . Find the electric field at the origin.
- 1–2 A spherically symmetric charge distribution has the following charge density profile

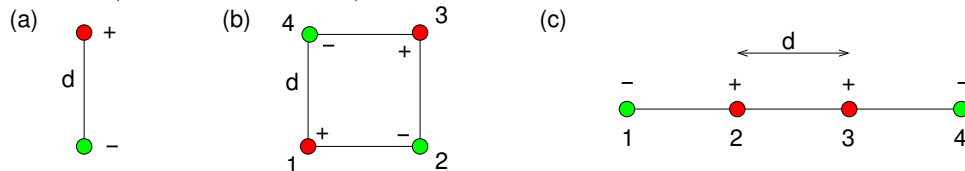
$$\rho(r, \theta, \phi) = \begin{cases} \rho_0 & (r < a) \\ \rho_0(r/a)^{-\beta} & (r \geq a) \end{cases} \quad (1.32)$$

where  $\beta$  is a constant ( $2 < \beta < 3$ ). Find the electric field and electrostatic potential everywhere.

- 1–3 The electric field is given by  $\mathbf{E}(\mathbf{r}) = E_0 \cos(z/z_0) \exp(-r/r_0) \hat{\mathbf{r}}$ , where  $z_0$  and  $r_0$  are constants. Find the charge density.
- 1–4 If we had a point charge  $q$  at the origin we might choose the reference point to be some point at an arbitrary distance  $r_0$  (usually infinity) from the origin. Then if we wish to find  $V(r, \theta, \phi)$  it would be convenient to have the reference point at  $\mathbf{r}_0 = (r_0, \theta, \phi)$ . Although obtaining the potential in this case is trivial, and one would usually just write it down, obtain the potential by carrying out explicitly the line integral for an appropriately parameterised curve.
- 1–5 The electric field is given by  $\mathbf{E}(\mathbf{r}) = E_0 \cos(z/z_0) \exp(-r/r_0) \hat{\mathbf{r}}$ , where  $z_0$  and  $r_0$  are constants. Check whether or not the electric field is conservative. If it is conservative find

the potential, if it isn't suggest how it may be possible to find the electrostatic part of the electric field (if present) and the corresponding electrostatic potential  $V(\mathbf{r})$ .

- 1-6 How much work must be done to assemble: (a) a physical dipole made of charge  $+q$  and charge  $-q$  separated by distance  $d$ , (b) a physical quadrupole made up of four charges  $+q$ ,  $-q$ ,  $+q$  and  $-q$  on successive corners of a square of side  $d$ , and (c) a physical quadrupole made up of four charges  $-q$ ,  $+q$ ,  $+q$  and  $-q$  equally spaced apart by distance  $d$  on a straight line (see diagram below).



- 1-7 (a) Use Gauss' law in integral form to find the electric field due to charge density  $\rho(\mathbf{r}) = \rho_0 \exp(-r/r_0)$ , and (b) check that you obtain the original charge density by taking the divergence of the electric field you find.
- 1-8 An isolated conducting sphere of radius  $a$  has net charge  $Q$ . Find how much work was done to charge the sphere using two different methods: (a) from the charge on the sphere and its potential, (b) by finding the energy stored in the electric field.

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## 2 Poisson's and Laplace's equations

### Learning objectives

- To be able to derive Poisson's equation from Gauss' law, and know that Laplace's equation is a special case of Poisson's equation which applies in charge-free regions.
- To know that the method of images for solving Poisson's equation works by substituting appropriate image charges for conducting boundaries, and to be able to obtain the potential for simple cases where there is plane, spherical and cylindrical symmetry.
- To be able to solve Laplace's equation in Cartesian coordinates in 3D for the potential inside a rectangular box where the potential or surface charge density is specified on all surfaces, and also for the case where the potential only depends on two of the coordinates.
- To be able to solve Laplace's equation in spherical coordinates inside and outside a spherical surface where the potential or surface charge density is specified, using the Legendre polynomials in the case of no azimuthal dependence of the boundary conditions, and using spherical harmonics for the general case.
- To learn how to use a finite-difference numerical method for solving Laplace's equation.

### 2.1 Poisson's equation

Named after Simon-Denis Poisson (1781–1840), Poisson's equation is obtained from Gauss' law by taking the divergence of  $\mathbf{E} = -\nabla V$ ,

$$\nabla \cdot (-\nabla V) = \frac{\rho(\mathbf{r})}{\epsilon_0} \quad (2.1)$$

giving *Poisson's equation*

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0}. \quad (2.2)$$

We already know its solution for the boundary condition  $V(r \rightarrow \infty) = 0$ , i.e. Eq. 1.20

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3r'. \quad (2.3)$$



As an aside, we can use Poisson's equation and the potential due to a point charge to check an identity involving the 3D Dirac delta function  $\delta^3(\mathbf{r})$  which is described in Appendix C.7. A point charge  $q$  located at  $\mathbf{r}_q$  is represented by charge density  $\rho(\mathbf{r}) = q\delta^3(\mathbf{r} - \mathbf{r}_q)$ . But for this case the potential is given by Eq. 1.19, so

$$\nabla^2 \left( \frac{q}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_q|} \right) = -\frac{q}{\epsilon_0}\delta^3(\mathbf{r} - \mathbf{r}_q), \quad (2.4)$$

$$\therefore \nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}_q|} = -4\pi\delta^3(\mathbf{r} - \mathbf{r}_q), \quad (2.5)$$

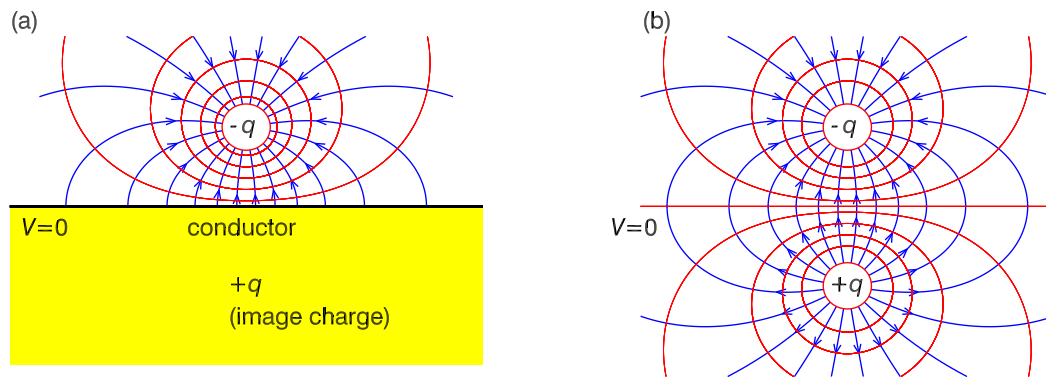
in agreement with Eq. C.38.

Often the boundary conditions differ from  $V(r \rightarrow \infty, \theta, \phi) = 0$ , and we will need to solve Poisson's equation or Laplace's equation taking account of boundary conditions determined by the problem. The usual types of boundary condition are *Dirichlet boundary conditions* for which  $V$  is specified at a boundary (e.g. a system of conductors held at various known potentials), and *Neumann boundary conditions* for which  $\partial V/\partial n|_{\text{boundary}}$  is specified at a boundary. Note that  $\partial V/\partial n$  represents the rate of increase of  $V$  with distance in the direction normal to the surface, and that  $\mathbf{E} \cdot \hat{\mathbf{n}} = -\partial V/\partial n$ . From Gauss' law, this is equivalent to specifying the surface charge density at the boundary surface. In Appendix E it is shown that the solution of Poisson's equation is unique if  $V$  is specified anywhere on the boundaries, or is unique up to an additive constant potential if  $\mathbf{E} \cdot \hat{\mathbf{n}}$  is specified on the boundaries.

## 2.2 Method of images

### 2.2.1 Point charge above a grounded plane conductor

The method of images is a powerful way of solving Poisson's equation where the symmetry of the problem permits. The simplest case is that of a charge  $-q$ , say, some distance away from a grounded plane conductor as shown in Fig. 2.1(a). If we have two equal but opposite charges in empty space then the potential in the plane midway between the charges will be zero (Fig. 2.1b), giving the same electric field above that plane as above the grounded conductor with the single charge  $-q$ . It is as if at the mirror-image point below the conducting plane there is another charge, the *image charge*,  $q_i$ , which in the case a plane conductor is the same magnitude but opposite sign to the real charge.



**Figure 2.1:** (a) Point charge  $-q$  above a grounded plane conductor showing field lines (blue curves with arrows), equipotential contours (red curves) and the location of the image charge. (b) Field lines and equipotentials of two real charges  $-q$  and  $+q$  in the same locations as the real charge and image charge in part (a).

As an example, consider the case of a grounded conductor in the  $xy$  plane with charge  $-q$  located at  $(0, 0, +z_0)$ . The potential and electric field for  $z > 0$  can be calculated as the sum of those due to the real charge  $-q$  and image charge  $+q$  at  $(0, 0, -z_0)$ ,



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$$V(x, y, z > 0) = \frac{-q}{4\pi\epsilon_0\sqrt{x^2 + y^2 + (z - z_0)^2}} + \frac{+q}{4\pi\epsilon_0\sqrt{x^2 + y^2 + (z + z_0)^2}}, \quad (2.6)$$

$$\mathbf{E}(x, y, z > 0) = \frac{-q}{4\pi\epsilon_0} \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + (z - z_0)\hat{\mathbf{z}}}{[x^2 + y^2 + (z - z_0)^2]^{3/2}} + \frac{+q}{4\pi\epsilon_0} \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + (z + z_0)\hat{\mathbf{z}}}{[x^2 + y^2 + (z + z_0)^2]^{3/2}}. \quad (2.7)$$

The resulting field lines and equipotential surfaces (equipotentials for short) are plotted in Fig. 2.1(a). Notice that at the surface of the conductor the electric field lines (and  $\mathbf{E}$ ) are normal to the surface. The reason for this is that electric field lines start or end on charges, and that if  $\mathbf{E}$  was not normal to the surface at some location, the charges on the surface would experience sideways forces redistributing them across the surface until  $\mathbf{E}$  became normal everywhere on the surface.

Surface charge will be present on the conducting surface, as we can easily show using Gauss' law. Imagine a Gaussian pillbox similar to that in Fig. 1.4(a) but with the important difference that the plane is the surface of a conductor and the volume below the plane (where the lower part of the pillbox is) is inside the conductor. Because in electrostatics  $\mathbf{E} = 0$  inside a conductor, there is only a contribution to  $\oint \mathbf{E} \cdot d\mathbf{S}$  from the upper surface of the pillbox, and the surface charge density on a conductor must therefore be in general

$$\sigma(\mathbf{r}) = \epsilon_0 \mathbf{E} \cdot \hat{\mathbf{n}} = \epsilon_0 (-\nabla V) \cdot \hat{\mathbf{n}} = -\epsilon_0 \frac{\partial V}{\partial n}, \quad (2.8)$$

where  $\partial V / \partial n$  is the directional derivative normal to the surface and is defined as  $(\nabla V) \cdot \hat{\mathbf{n}}$ .

Returning to the example above, at the surface  $z = 0$ , and so the electric field from Eq. 2.7 becomes

$$\mathbf{E}(x, y, 0) = \frac{q}{2\pi\epsilon_0} \frac{z_0 \hat{\mathbf{z}}}{[x^2 + y^2 + z_0^2]^{3/2}}, \quad (2.9)$$

and hence the surface charge density on the conductor is

$$\sigma(x, y, 0) = \frac{q}{2\pi} \frac{z_0}{(x^2 + y^2 + z_0^2)^{3/2}} = \frac{q}{2\pi} \frac{z_0}{(s^2 + z_0^2)^{3/2}} \equiv \sigma(s, z_0) \quad (2.10)$$

where  $s = (x^2 + y^2)^{1/2}$ . Notice that the induced surface charge density in this case is positive everywhere, and this is because the charge near the conductor which induced the surface charge is negative.

We can integrate the surface charge density over the conductor to find the total induced surface charge,

$$q_s = \int_0^\infty 2\pi s \sigma(s, z_0) ds = \frac{q}{2\pi} \int_0^\infty 2\pi s \frac{z_0}{(s^2 + z_0^2)^{3/2}} dR = -q \left[ \frac{z_0}{\sqrt{s^2 + z_0^2}} \right]_0^\infty = +q. \quad (2.11)$$

Finally, the force between charge  $-q$  and the grounded plane must be the same as would exist between the real charge  $-q$  and a real charge of magnitude equal to  $q_i$  (in this case  $+q$ ) at the image charge's location

$$F = \frac{1}{4\pi\epsilon_0} \frac{qq_i}{R_{qq_i}^2} = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{(2z_0)^2} \quad (2.12)$$

where  $R_{qq_i}$  is the distance between the image charge and the real charge, in this case  $R_{qq_i} = 2z_0$ .

### 2.2.2 Point charge outside grounded spherical conductor

The method of images may be used to find the potential due to a point charge  $q$  outside a grounded conducting sphere of radius  $a$  (as in Fig. 2.2a) by replacing the sphere by a fictitious image charge,  $q_i$ , inside the conductor. From symmetry arguments, the image charge  $q_i$  must be on the line from sphere's center to charge  $q$ , and the sum of the potentials due to the real charge and image charge must be zero on the surface of the sphere, e.g. at point P,

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left( \frac{q}{d} + \frac{q_i}{b} \right), \quad (2.13)$$

$$= \frac{1}{4\pi\epsilon_0} \left( \frac{q}{\sqrt{a^2 + z_q^2 - 2az_q \cos \theta}} + \frac{q_i}{\sqrt{a^2 + z_i^2 - 2az_i \cos \theta}} \right), \quad (2.14)$$

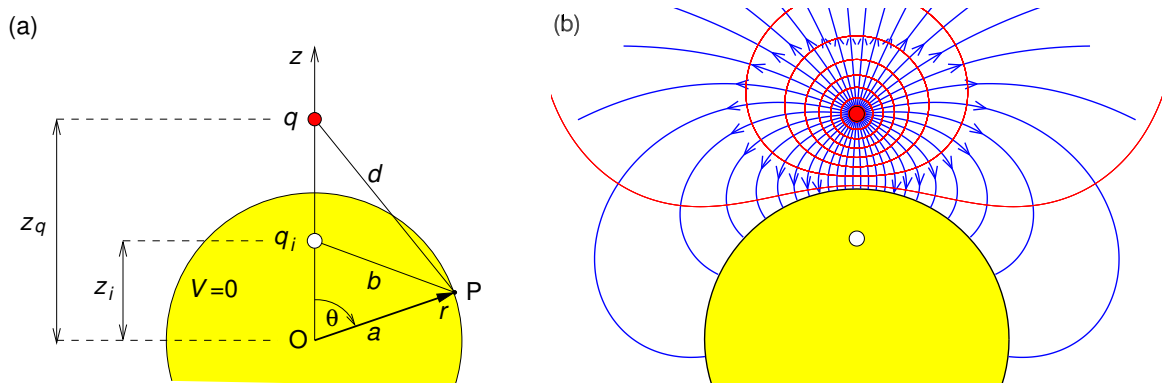
$$= \frac{1}{4\pi\epsilon_0} \left( \frac{q/a}{\sqrt{1 + (z_q/a)^2 - 2(z_q/a) \cos \theta}} + \frac{q_i/z_i}{\sqrt{1 + (a/z_i)^2 - 2(a/z_i) \cos \theta}} \right). \quad (2.15)$$

By inspection, the boundary condition  $V = 0$  on the surface of the sphere is satisfied for any  $\theta$ -value by  $q/a = -q_i/z_i$  and  $z_q/a = a/z_i$ , from which we obtain the location and magnitude of

the image charge

$$z_i = a^2/z_q, \quad q_i = -(a/z_q)q. \quad (2.16)$$

The resulting field lines and equipotentials are plotted in Fig. 2.2(b).



**Figure 2.2:** Point charge outside grounded spherical conductor showing (a) position of image charge, and (b) field lines and equipotentials.

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### 2.2.3 Line charge outside grounded cylindrical conductor

We previously used Gauss' law to find  $\mathbf{E}$  due to an infinite line charge (Fig. 1.4b, and Eq. 1.14)

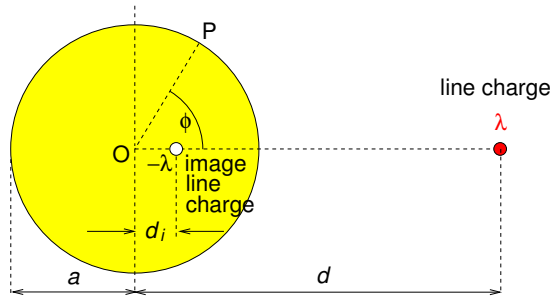
$$\mathbf{E} = \frac{\lambda}{2\pi\rho\epsilon_0}\hat{\rho}. \quad (2.17)$$

From the electric field we find the potential

$$V(\mathbf{r}) = - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}' \quad (2.18)$$

where  $\mathbf{r}_0$  is the reference point where the potential is defined to be 0. Because of the cylindrical symmetry about the line charge, we can define the reference “point” to be anywhere on a concentric cylinder of arbitrary cylindrical radius  $\rho_0$ , and so the potential is

$$V(\mathbf{r}) = - \int_{\rho_0}^{\rho} \frac{\lambda}{2\pi\rho\epsilon_0} \hat{\rho} \cdot (\hat{\rho} d\rho) = - \frac{\lambda}{2\pi\epsilon_0} \int_{\rho_0}^{\rho} \frac{d\rho}{\rho} = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{\rho_0}{\rho}\right). \quad (2.19)$$



**Figure 2.3:** Line charge outside grounded cylindrical conductor showing position of image line charge, and geometry for finding the potential.

Next consider the potential due to two parallel line charges  $\lambda_1$  and  $\lambda_2$ . Suppose that for line charge 1 the reference “point” is arbitrarily set at cylindrical radius  $\rho_0^{(1)}$  away from it, and for line charge 2 the reference “point” is arbitrarily set at cylindrical radius  $\rho_0^{(2)}$  away from it. Clearly the two reference “points” can not coincide, and so one should ask whether or not this matters, but as we shall see it doesn't. Let's just go ahead and apply the principle of



superposition and write down the potential due to the two line charges

$$V(\mathbf{r}) = \frac{\lambda_1}{2\pi\epsilon_0} \ln\left(\frac{\rho_0^{(1)}}{\rho_1}\right) + \frac{\lambda_2}{2\pi\epsilon_0} \ln\left(\frac{\rho_0^{(2)}}{\rho_2}\right) \quad (2.20)$$

$$= \frac{1}{2\pi\epsilon_0} \left( \lambda_1 \ln \rho_0^{(1)} + \lambda_2 \ln \rho_0^{(2)} \right) - \frac{1}{2\pi\epsilon_0} (\lambda_1 \ln \rho_1 + \lambda_2 \ln \rho_2) \quad (2.21)$$

$$= V_0 - \frac{1}{2\pi\epsilon_0} (\lambda_1 \ln \rho_1 + \lambda_2 \ln \rho_2). \quad (2.22)$$

where  $\rho_1$  is the distance of line charge 1 to the field point  $\mathbf{r}$ , and  $\rho_2$  is the distance of line charge 2 to the field point. Note that in Eq. 2.21 the first term involves only constants, the constant line charge densities  $\lambda_1$  and  $\lambda_2$  and the arbitrary constants  $\rho_0^{(1)}$  and  $\rho_0^{(2)}$ , and so may be written as an arbitrary constant potential  $V_0$  whose value will not affect the electric field or the potential difference between two points.

Equipotential contours and field lines are plotted in Fig. 2.4(a) for the the case of two parallel line charges with equal but opposite line charge densities,  $+\lambda$  and  $-\lambda$ , for which

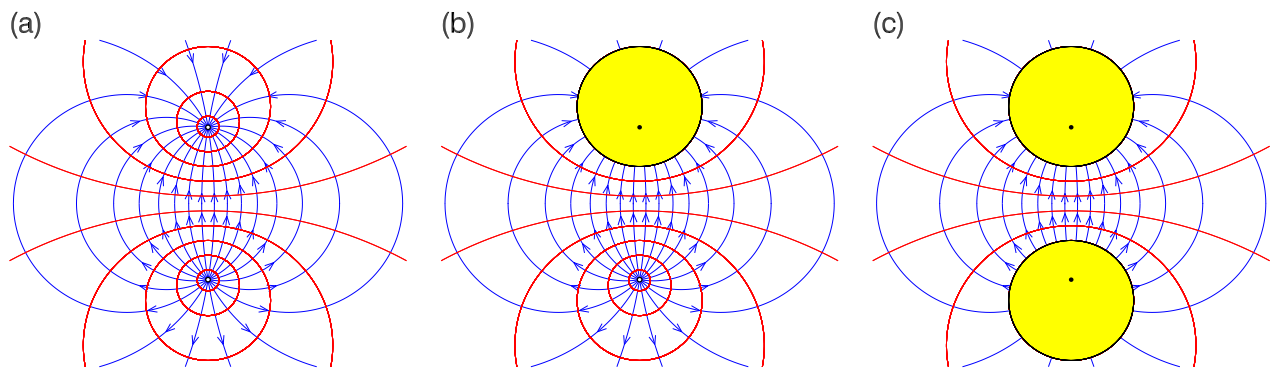
$$V(\mathbf{r}) = V_0 - \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{\rho_1}{\rho_2}\right). \quad (2.23)$$

Notice that the equipotential surfaces for this case are of circular cross section. This tells us immediately that for the case of a line charge and a parallel grounded conducting cylinder we can use the method of images with an equal but opposite image line charge inside the conducting cylinder and parallel to the real line charge.


In Fig. 2.3 The line charge density  $\lambda$  ( $\text{C m}^{-1}$ ) is parallel to the conducting cylinder's axis and distance  $d$  from it. We have just seen that we can use an image line charge  $-\lambda$  which from symmetry must be parallel to to and in the plane containing the cylinder axis and the real line charge  $\lambda$ . One can show (Exercise 2—3) that the cylinder's surface is an equipotential if

$$d_i = a^2/d. \quad (2.24)$$

Equipotential contours and electric field lines are shown in Fig. 2.4(b). The result above may also be applied to the case of two parallel conducting cylinders, e.g. a twin wire transmission line having equal but opposite charges (Fig. 2.4c) in order to work out the capacitance per unit length (Exercise 2–4); the capacitance of two conductors with potential difference  $V$  and having charge  $+q$  on one and  $-q$  on the other is  $C = q/V$  and will be discussed further in Chapter 4.



**Figure 2.4:** Electric field lines and equipotentials around: (a) two parallel line charges  $+\lambda$  and  $-\lambda$ , (b) one line charge  $+\lambda$  and a parallel conducting cylinder, (c) two parallel conducting cylinders with equal and opposite charge. In (b) and (c) the dots show the positions of image charges.



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

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## 2.3 Laplace's Equation

Named after Pierre-Simon, Marquis de Laplace (1749–1827), Laplace's equation is simply Poisson's equation for  $\rho(\mathbf{r}) = 0$ , and in the three main coordinate systems is given by

$$\text{Cartesian:} \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \quad (2.25)$$

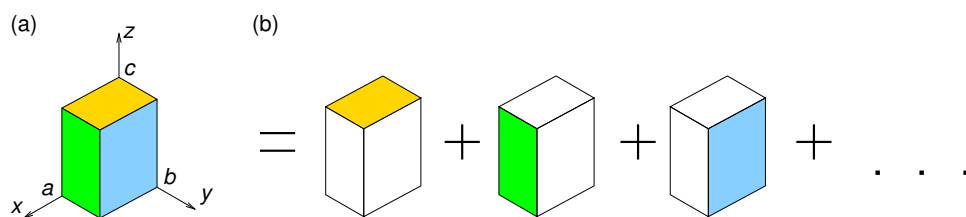
$$\text{Cylindrical:} \quad \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial V}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0. \quad (2.26)$$

$$\text{Spherical:} \quad V \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0. \quad (2.27)$$

It must be solved together with boundary conditions appropriate to the problem.

### 2.3.1 3D Laplace's equation in Cartesian coordinates

Using the method of separation of variables we will find the general solution for the potential inside a rectangular box where the potential is specified to be zero on all but one of its surfaces. The particular solution can then be obtained by applying the boundary condition that the potential must be that specified on the surface which has  $V \neq 0$ . Having solved for this case, we can do the same for  $V$  being non-zero on each of the other five sides, and then use the principle of superposition to get the full solution. The geometry to be used for solving Laplace's equation is shown in Fig. 2.5(a) together with a cartoon (Fig. 2.5b) indicating how the full solution may be built up from six solutions, each being for when the potential is specified on one of the surfaces and is zero on the other five.



**Figure 2.5:** (a) Geometry for solving Laplace's equation in 3D in Cartesian coordinates for the field inside a rectangular box where the potential is specified on the six surfaces. (b) A cartoon indicating how the full solution may be built up from six solutions.

Using the method of separation of variables we try to find a solution of the form  $V(x, y, z) =$

$X(x)Y(y)Z(z)$  and substitute this into Laplace's equation, which becomes

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) X(x)Y(y)Z(z) = 0, \quad (2.28)$$

$$\frac{d^2 X(x)}{dx^2} Y(y)Z(z) + X(x) \frac{d^2 Y(y)}{dy^2} Z(z) + X(x)Y(y) \frac{d^2 Z(z)}{dz^2} = 0, \quad (2.29)$$

$$\therefore \frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \frac{Z''(z)}{Z(z)} = 0. \quad (2.30)$$

Note that the 1st term depends only on  $x$ , the 2nd only on  $y$  and the 3rd only on  $z$ . The only way that the equation can hold as  $x$ ,  $y$  or  $z$  are varied independently is if each term is a constant, and the three *separation constants* add to zero,

$$\frac{X''(x)}{X(x)} = S_1, \quad \frac{Y''(y)}{Y(y)} = S_2, \quad \frac{Z''(z)}{Z(z)} = S_3, \quad \text{where } S_1 + S_2 + S_3 = 0. \quad (2.31)$$

At least one of the three separation constants must be positive and at least one must be negative.

For our hollow rectangular box (Fig. 2.5) let's choose the surface with non-zero potential to be that at  $z = c$ , on which the potential is specified to be  $V(x, y, c) = V_0(x, y)$ . It makes sense to have  $S_1$  and  $S_2$  negative so that the equations in  $x$  and in  $y$  will have periodic solutions, making it easy to satisfy their boundary conditions. Then the three equations to solve are

$$X''(x) = -\alpha^2 X(x), \quad Y''(y) = -\beta^2 Y(y), \quad Z''(z) = \gamma^2 Z(z), \quad (2.32)$$

where  $\gamma^2 = \alpha^2 + \beta^2$ . These equations have well-known solutions

$$X(x) = a_1 \sin(\alpha x) + a_2 \cos(\alpha x), \quad (2.33)$$

$$Y(y) = b_1 \sin(\beta y) + b_2 \cos(\beta y), \quad (2.34)$$

$$Z(z) = c_1 \exp(\gamma z) + c_2 \exp(-\gamma z), \quad (2.35)$$

as can easily be checked by substitution. For Eqs. 2.33 and 2.34 the boundary conditions, i.e.  $V(0, y, z) = V(a, y, z) = 0$  and  $V(x, 0, z) = V(x, b, z) = 0$ , require  $a_2 = b_2 = 0$  and that

$$\alpha = \alpha_k \equiv \frac{k\pi}{a}, \quad \beta = \beta_l \equiv \frac{l\pi}{b}, \quad \text{and so } \gamma^2 = \gamma_{kl}^2 \equiv \alpha_k^2 + \beta_l^2, \quad (2.36)$$

where  $k$  and  $l$  are integers. For Eq. 2.35 the boundary condition  $V(x, y, 0) = 0$  requires  $c_2 = -c_1$ ,

and so  $Z(z) \propto (e^{\gamma z} - e^{-\gamma z})/2 = \sinh(\gamma z)$ ; actually, we could equally well have written the general solution of Eq. 2.35 in terms of the hyperbolic sine and cosine functions (Fig. 2.6), i.e.

$$Z(z) = d_1 \sinh(\gamma z) + d_2 \cosh(\gamma z) = d_1 \frac{e^{\gamma z} - e^{-\gamma z}}{2} + d_2 \frac{e^{\gamma z} + e^{-\gamma z}}{2}. \quad (2.37)$$

A useful property of the hyperbolic sine and cosine functions is that

$$\frac{d}{dz} \sinh(z) = \cosh(z) \quad \text{and} \quad \frac{d}{dz} \cosh(z) = \sinh(z). \quad (2.38)$$

Then, the general solution to Laplace's Equation for the case where  $V(x, y, c) \neq 0$  with the potential on all other surfaces being zero is

$$V(x, y, z) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} A_{kl} \sinh(\gamma_{kl} z) \sin(\alpha_k x) \sin(\beta_l y). \quad (2.39)$$

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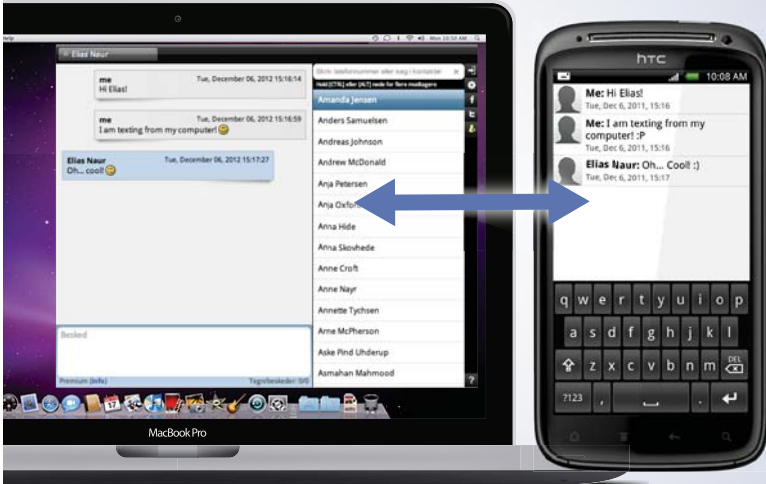
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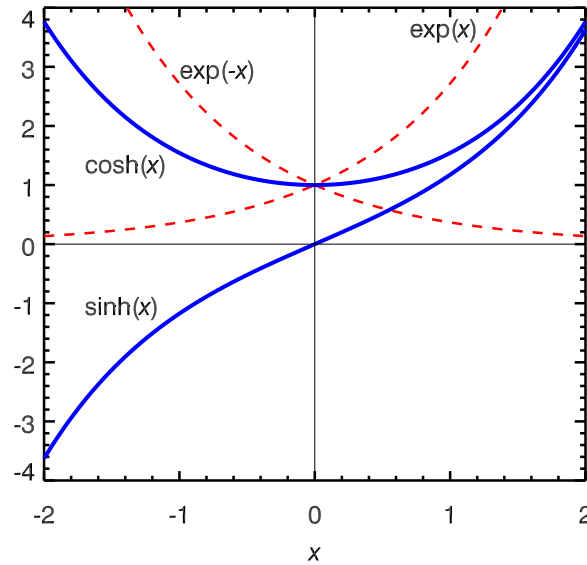
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**Figure 2.6:** Hyperbolic sine and cosine functions compared with the positive and negative exponentials.

The coefficients  $A_{kl}$  can be obtained using the boundary condition at  $z = c$ , i.e. by solving

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} A_{kl} \sinh(\gamma_{kl} c) \sin(\alpha_k x) \sin(\beta_l y) = V_0(x, y). \quad (2.40)$$

The easiest way to solve this is by recognising that the left hand side of Eq. 2.40 is actually a 2D Fourier sine series, i.e.

$$V_0(x, y) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} B_{kl} \sin\left(\frac{k\pi x}{a}\right) \sin\left(\frac{l\pi y}{b}\right), \quad (2.41)$$

for which the Fourier series coefficients are

$$B_{kl} = \frac{4}{ab} \int_0^a \int_0^b V_0(x, y) \sin\left(\frac{k\pi x}{a}\right) \sin\left(\frac{l\pi y}{b}\right) dx dy. \quad (2.42)$$

Hence

$$A_{kl} = \frac{B_{kl}}{\sinh(\gamma_{kl} c)} = \frac{4}{a b \sinh(\gamma_{kl} c)} \int_0^a \int_0^b V_0(x, y) \sin(\alpha_k x) \sin(\beta_l y) dx dy. \quad (2.43)$$

### 2.3.2 2D Laplace's Equation in Cartesian Coordinates

If there is no dependence on  $z$ , we need only to solve Laplace's equation in 2D for the region inside a rectangular cross-section pipe. Again the method of separation of variables can be used together with the principle of superposition to break down a problem with  $V \neq 0$  on all four sides to 4 simpler problems with  $V \neq 0$  only on one side. We need to decide after examining the problem whether it is sensible to have the periodic solution for  $X(x)$  or for  $Y(y)$ . Keep the negative sign for the equation for  $X(x)$  if the boundary conditions suggest it would be simpler to solve if we have a periodic solution in  $x$ , e.g. if  $V=0$  at both boundaries in  $x$ .

The general solutions are then of the form

$$V(x, y) = (A \cos(kx) + B \sin(kx)) (Ce^{ky} + De^{-ky}), \quad (2.44)$$

or equivalently

$$V(x, y) = (A \cos(kx) + B \sin(kx)) [C' \cosh(ky) + D' \sinh(ky)]. \quad (2.45)$$

Finally, the boundary conditions are applied, as for the 3D case, to find allowed values of  $k$  and the coefficients in the Fourier series for the potential on the side where  $V \neq 0$ , and so the full solution. For example, inside the rectangular region corresponding to  $0 < x < a$  and  $0 < y < b$  with  $V(x, b) = V_0(x)$ , and  $V=0$  on the other 3 sides,

$$V(x, y) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right), \quad (2.46)$$

and the boundary condition at  $y = b$  determines the coefficients  $A_n$  in this Fourier sine series

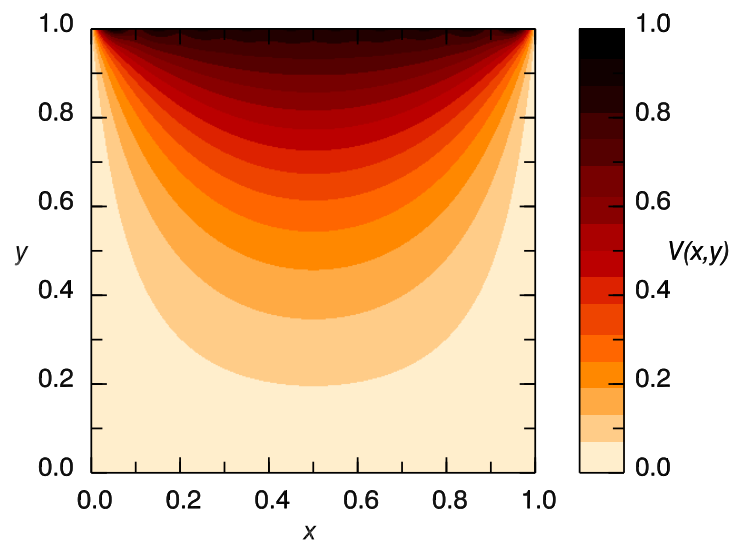
$$A_n = \frac{2}{a} \frac{1}{\sinh\left(\frac{n\pi}{a}b\right)} \int_0^a V_0(x) \sin\left(\frac{n\pi}{a}x\right) dx. \quad (2.47)$$

As an example we shall consider the case where  $a = b = 1$  m,  $V(x, b) = 1$  V and  $V = 0$  on the other three sides. Then,

$$A_n = \frac{2}{\sinh(n\pi)} \int_0^1 \sin(n\pi x) dx. \quad \therefore A_n = \frac{4}{n\pi \sinh(n\pi)} \quad (\text{for odd } n). \quad (2.48)$$



Using Eqs. 2.46 and 2.48, and summing terms up to  $n = 19$  we obtain the solution contoured in Fig. 2.7.



**Figure 2.7:** Contours of  $V$  for the the result using Eqs. 2.46 and 2.48, and summing terms up to  $n = 19$ .

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We can obtain the electric field from the potential in the usual way

$$E_x = -\frac{\partial V}{\partial x} = -4 \sum_{n, \text{ odd}} \frac{1}{\sinh(n\pi)} \cos(n\pi x) \sinh(n\pi y), \quad (2.49)$$

$$E_y = -\frac{\partial V}{\partial y} = -4 \sum_{n, \text{ odd}} \frac{1}{\sinh(n\pi)} \sin(n\pi x) \cosh(n\pi y). \quad (2.50)$$

Another example of solving Laplace's equation in 2D is given in Exercise 2–5.

## 2.4 Laplace's equation in spherical coordinates

In spherical coordinates Laplace's equation is

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0. \quad (2.51)$$

### 2.4.1 Solution for no azimuthal dependence

Here we will first consider the case of azimuthal symmetry ( $\partial V / \partial \phi = 0$ ) and look for solutions of the form  $V(r, \theta) = R(r)\Theta(\theta)$ . Then

$$\frac{1}{R(r)} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) = - \frac{1}{\Theta(\theta) \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta(\theta)}{d\theta} \right). \quad (2.52)$$

The left hand side depends only on  $r$ , and the right hand side depends only on  $\theta$ . Each side must equal the same separation constant  $C$ , and so we have two ordinary differential equations

$$\frac{1}{R(r)} \frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) = C, \quad (2.53)$$

$$- \frac{1}{\Theta(\theta) \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta(\theta)}{d\theta} \right) = C. \quad (2.54)$$

We will first tackle the radial equation (Eq. 2.53), and to simplify later working (we'll see

shortly) set  $C \equiv \alpha(\alpha + 1)$  and write it as

$$\frac{d}{dr} \left( r^2 \frac{dR(r)}{dr} \right) = \alpha(\alpha + 1)R(r). \quad (2.55)$$

This equation has general solution

$$R(r) = Ar^\alpha + Br^{-(\alpha+1)} \quad (2.56)$$

as we can easily check by substitution.

The angular equation (Eq. 2.54) can be re-written in terms of  $\cos \theta$ ,

$$\frac{d}{d(\cos \theta)} \left[ (1 - \cos^2 \theta) \frac{d\Theta}{d(\cos \theta)} \right] + \alpha(\alpha + 1)\Theta = 0, \quad (2.57)$$

which is known as *Legendre's equation*. This equation can be solved by a power series method, and the series converges for  $-1 \leq \cos \theta \leq 1$  for integer values of  $\alpha = \ell \geq 0$ . Since Legendre's equation is 2nd order, there are two linearly-independent solutions for each value of  $\alpha$ , of which one is infinite for  $\cos \theta = \pm 1$ . The finite solutions are called “Legendre polynomials”,  $P_\ell(\cos \theta)$ , and the first 6 Legendre polynomials are

$$P_0(\cos \theta) = 1, \quad (2.58)$$

$$P_1(\cos \theta) = \cos \theta, \quad (2.59)$$

$$P_2(\cos \theta) = \frac{1}{2}(3 \cos^2 \theta - 1), \quad (2.60)$$

$$P_3(\cos \theta) = \frac{1}{2}(5 \cos^3 \theta - 3 \cos \theta), \quad (2.61)$$

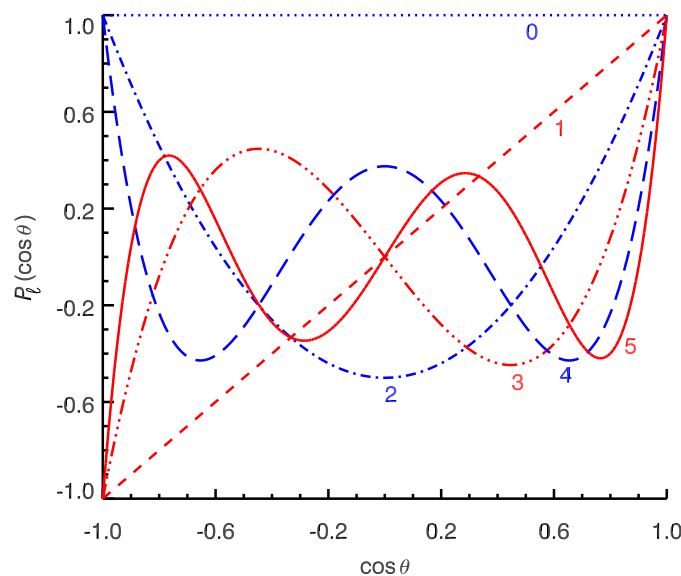
$$P_4(\cos \theta) = \frac{1}{8}(34 \cos^4 \theta - 30 \cos^2 \theta + 3), \quad (2.62)$$

$$P_5(\cos \theta) = \frac{1}{8}(63 \cos^5 \theta - 70 \cos^3 \theta + 15 \cos \theta), \quad (2.63)$$

and are plotted in Fig. 2.8. Notice that  $P_\ell(\cos \theta)$  are normalised to 1 at  $\cos \theta = 1$ , and that  $P_\ell(\cos \theta)$  is an even function if  $\ell$  is even, and an odd function if  $\ell$  is odd.

The Legendre polynomials form a complete orthogonal set

$$\int_{-1}^1 P_{\ell'}(\cos \theta) P_\ell(\cos \theta) d \cos \theta = \frac{2}{2\ell + 1} \delta_{\ell \ell'}, \quad (2.64)$$



**Figure 2.8:** The first 6 Legendre polynomials.

where  $\delta_{ij}$  is the *Kronecker delta* which takes the value 1 if  $i = j$  and 0 if  $i \neq j$ . Note the

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similarity to the sine and cosine functions in Fourier series; this enables us to represent any function  $f(x)$  over the range  $-1 \leq x \leq +1$  as a sum of Legendre polynomials

$$f(x) = \sum_{\ell=0}^{\infty} C_{\ell} P_{\ell}(x), \quad \text{where } C_{\ell} = \frac{2\ell+1}{2} \int_{-1}^1 f(x) P_{\ell}(x) dx. \quad (2.65)$$

If  $f(x)$  is an even (odd) function, only terms with even (odd)  $\ell$  will be non-zero.

The General solution of Laplace's equation with axial symmetry is then

$$V(r, \theta) = \sum_{\ell=0}^{\infty} (A_{\ell} r^{\ell} + B_{\ell} r^{-(\ell+1)}) P_{\ell}(\cos \theta). \quad (2.66)$$

The coefficients in the series are obtained from the boundary conditions. If the region inside the boundaries contains  $r = 0$  then all  $B_{\ell}$  will be zero, but if the region inside the boundaries contains  $r = \infty$  then (usually) all  $A_{\ell}$  will be zero. At the boundary (often a spherical surface with radius  $r = a$ ) where the potential is specified, we can use the orthogonality of  $P_{\ell}(\cos \theta)$  to find the coefficients. Hence, for the region inside a spherical surface of radius  $r = a$ , where  $V(a, \cos \theta) = V_0(\cos \theta)$ , we have

$$A_{\ell} a^{\ell} = \frac{2\ell+1}{2} \int_{-1}^1 V_0(\cos \theta) P_{\ell}(\cos \theta) d \cos \theta. \quad (2.67)$$

Often it is simpler to re-write  $V_0(\cos \theta)$  as a series of Legendre polynomials and obtain  $A_{\ell}$  simply by comparing it with  $V(a, \cos \theta)$  in Eq. 2.66. An example is given in Exercise 1–7.

## 2.4.2 Solution including azimuthal dependence

In this case the potential depends on both  $\theta$  and  $\phi$  at a spherical boundary. We will represent the angular dependence in terms of *spherical harmonics*

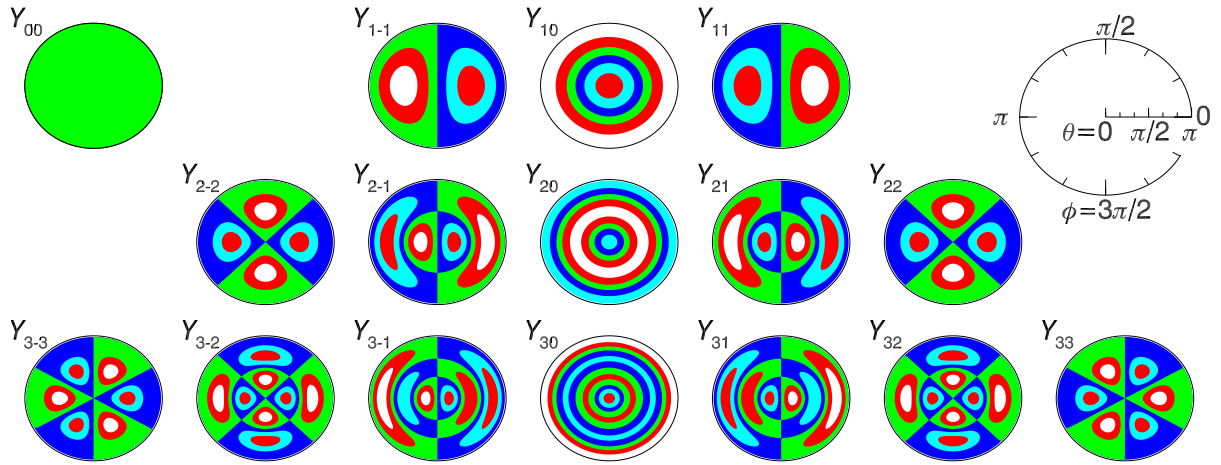
$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{(2\ell+1)(\ell-1)!}{4\pi(\ell+1)!}} P_{\ell}^m(\cos \theta) e^{im\phi}, \quad (2.68)$$

where

$$P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x) \quad (2.69)$$

are *associated Legendre functions*. The first 9 spherical harmonics are given in Eq. 2.70, and the first 16 spherical harmonics are contoured in Fig. 2.9.

$$\begin{aligned} Y_{0,0} &= \frac{1}{\sqrt{4\pi}}. \\ Y_{1,0} &= \sqrt{\frac{3}{4\pi}} \cos \theta, \quad Y_{1,\pm 1} = \mp \sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm i\phi}. \\ Y_{2,0} &= \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1), \quad Y_{2,\pm 1} = \mp \sqrt{\frac{15}{8\pi}} \cos \theta \sin \theta e^{\pm i\phi}, \quad Y_{2,\pm 2} = \sqrt{\frac{15}{32\pi}} \sin^2 \theta e^{\pm 2i\phi}. \end{aligned} \quad (2.70)$$



**Figure 2.9:** The first 16 spherical harmonics. Note: these polar plots are not equal-area projections.

The spherical harmonics form a complete orthonormal set

$$\int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta Y_{\ell,m}^*(\theta, \phi) Y_{\ell',m'}(\theta, \phi) = \delta_{\ell,\ell'} \delta_{m,m'}, \quad (2.71)$$

and since we are integrating over all  $\theta$  and  $\phi$  we are actually integrating over all solid angles, i.e.

$$\oint d\Omega Y_{\ell,m}^*(\theta, \phi) Y_{\ell',m'}(\theta, \phi) = \delta_{\ell,\ell'} \delta_{m,m'}. \quad (2.72)$$

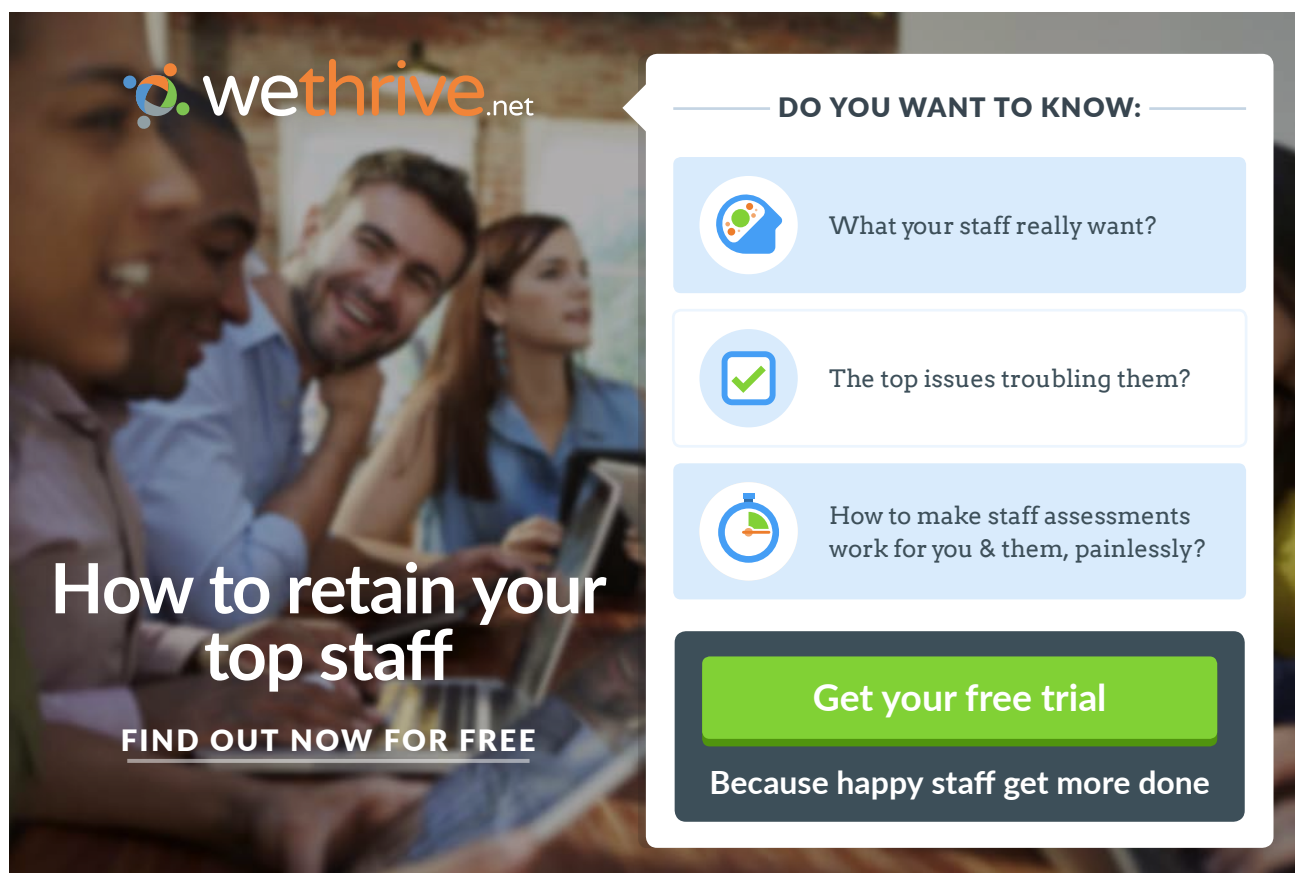
Using spherical harmonics, the general solution of Laplace's equation in spherical coordinates where there is azimuthal dependence is a straightforward extension of the case with no dependence on  $\phi$ ,

$$V(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (A_{\ell,m} r^{\ell} + B_{\ell,m} r^{-(\ell+1)}) Y_{\ell,m}(\theta, \phi). \quad (2.73)$$

Suppose, then that there is a spherical surface of radius  $a$  centred on the origin on which the potential is specified as  $V(a, \theta, \phi) = V_0(\theta, \phi)$ . If we wish to find the potential outside the sphere, to keep  $V$  finite all  $A_{\ell,m}$  must be zero in Eq. 2.73, and using Eq. 2.72 together with the boundary condition we obtain

$$V(r \geq a, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} B_{\ell,m} r^{-(\ell+1)} Y_{\ell,m}(\theta, \phi), \quad (2.74)$$

$$V_0(\theta, \phi) = \sum_{\ell'=0}^{\infty} \sum_{m'=-\ell'}^{\ell'} B_{\ell',m'} a^{-(\ell'+1)} Y_{\ell',m'}(\theta, \phi), \quad (2.75)$$



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$$\oint d\Omega Y_{\ell,m}^*(\theta, \phi) V_0(\theta, \phi) = \sum_{\ell'=0}^{\infty} \sum_{m'=-\ell'}^{\ell'} B_{\ell',m'} a^{-(\ell'+1)} \oint d\Omega Y_{\ell,m}^*(\theta, \phi) Y_{\ell',m'}(\theta, \phi), \quad (2.76)$$

$$\therefore \oint d\Omega Y_{\ell,m}^*(\theta, \phi) V_0(\theta, \phi) = \sum_{\ell'=0}^{\infty} \sum_{m'=-\ell'}^{\ell'} B_{\ell',m'} a^{-(\ell'+1)} \delta_{\ell,\ell'} \delta_{mm'}. \quad (2.77)$$

$$\therefore B_{\ell,m} = a^{(\ell+1)} \oint d\Omega Y_{\ell,m}^*(\theta, \phi) V_0(\theta, \phi). \quad (2.78)$$

Similarly, inside the sphere

$$V(r \leq a, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell,m} r^{\ell} Y_{\ell,m}(\theta, \phi), \quad (2.79)$$

$$A_{\ell,m} = a^{-\ell} \oint d\Omega Y_{\ell,m}^*(\theta, \phi) V_0(\theta, \phi). \quad (2.80)$$

As an example suppose the potential on the surface of a sphere is

$$V(a, \theta, \phi) = V_1 \sin \theta (1 + 2 \cos \theta) \sin \phi, \quad (2.81)$$

and we need to find the potential outside. This relatively simple boundary condition can be re-written in terms of spherical harmonics as follows

$$V(a, \theta, \phi) = V_1 \left[ \sqrt{\frac{2\pi}{3}} [Y_{1,1}(\theta, \phi) - Y_{1,-1}(\theta, \phi)] + 2\sqrt{\frac{2\pi}{15}} [Y_{2,1}(\theta, \phi) - Y_{2,-1}(\theta, \phi)] \right]. \quad (2.82)$$

Hence,

$$B_{1,1} = V_1 \sqrt{\frac{2\pi}{3}} a^2, \quad B_{1,-1} = -V_1 \sqrt{\frac{2\pi}{3}} a^2, \quad B_{2,1} = V_1 \sqrt{\frac{2\pi}{15}} a^3, \quad \text{and} \quad B_{2,-1} = -V_1 \sqrt{\frac{2\pi}{15}} a^3,$$

giving

$$\begin{aligned}
 V(r, \theta, \phi) &= V_1 \frac{a^2}{r^2} \sqrt{\frac{2\pi}{3}} [Y_{1,1}(\theta, \phi) - Y_{1,-1}(\theta, \phi)] + 2V_1 \frac{a^3}{r^3} \sqrt{\frac{2\pi}{15}} [Y_{121}(\theta, \phi) - Y_{2,-1}(\theta, \phi)], \\
 &= V_1 \frac{a^2}{r^2} \sin \theta \cos \phi + 2V_1 \frac{a^3}{r^3} \sin \theta \cos \theta \cos \phi.
 \end{aligned} \tag{2.83}$$

## 2.5 Finite-difference method for Laplace's equation

Numerical finite-difference methods may be used when analytic methods are impractical. They can be motivated by the equation of heat conduction in solids, which becomes Laplace's equation for heat-source free regions in equilibrium.

The equation of heat conduction in solids in a region where there are no heat sources is

$$\nabla^2 T = \frac{1}{\alpha} \frac{\partial T}{\partial t} \tag{2.84}$$

where  $T(\mathbf{r}, t)$  is the temperature and  $\alpha$  is the thermal diffusivity. If the boundaries are held at fixed temperatures then for any initial temperature distribution the temperature distribution evolves to a steady state at  $t \rightarrow \infty$  at which time

$$\nabla^2 T = 0, \tag{2.85}$$

which is Laplace's equation.

By analogy, in electrostatics we can solve Laplace's equation by numerically "evolving" an initial potential in "time" to  $t \rightarrow \infty$  using

$$\frac{\partial V}{\partial t} = \nabla^2 V, \tag{2.86}$$

or, in practice, until  $V(\mathbf{r})$  stops changing. The potential will then have "relaxed" to a steady-state and obey Laplace's equation. This equation *does not* represent the time-dependence of  $V(\mathbf{r})$  — we are still discussing *electrostatics*!



We shall solve Eq. 2.86 numerically by taking “time” steps  $\delta t$

$$V(\mathbf{r}, t + \delta t) = V(\mathbf{r}, t) + [\nabla^2 V(\mathbf{r}, t)] \delta t, \quad (2.87)$$

calculating  $V(\mathbf{r}, t)$  only at points on a 3D rectangular grid with spacing  $L$  and a “time” grid with spacing  $\delta t$ . We shall obtain  $\nabla^2 V$  by numerical differentiation, using the shorthand notation

$$V_{i,j,k}^n \equiv V(x = iL, y = jL, z = kL, t = n\delta t), \quad (2.88)$$

and carrying out numerical differentiation with respect to  $x$

$$\left. \frac{\partial V}{\partial x} \right|_{(x+L/2, y, z, t)} \approx \frac{V_{i+1, j, k}^n - V_{i, j, k}^n}{L}, \quad \left. \frac{\partial V}{\partial x} \right|_{(x-L/2, y, z, t)} \approx \frac{V_{i, j, k}^n - V_{i-1, j, k}^n}{L}, \quad (2.89)$$

$$\therefore \frac{\partial^2 V}{\partial x^2}(x, y, z, t) \approx \frac{V_{i+1, j, k}^n + V_{i-1, j, k}^n - 2V_{i, j, k}^n}{L^2}. \quad (2.90)$$

Hence, we obtain the Laplacian of  $V$  in 2D and in 3D,

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$$\begin{aligned} [\nabla^2 V]_{i,j}^n &\approx \frac{1}{L^2} (V_{i+1,j}^n + V_{i-1,j}^n + V_{i,j+1}^n + V_{i,j-1}^n - 4V_{i,j}^n), \\ [\nabla^2 V]_{i,j,k}^n &\approx \frac{1}{L^2} (V_{i+1,j,k}^n + V_{i-1,j,k}^n + V_{i,j+1,k}^n + V_{i,j-1,k}^n + V_{i,j,k+1}^n + V_{i,j,k-1}^n - 6V_{i,j,k}^n). \end{aligned} \quad (2.91)$$

Then a suitable iteration scheme in 2D would be

$$V_{i,j}^{n+1} = V_{i,j}^n + [\nabla^2 V]_{i,j}^n \delta t. \quad (2.92)$$

$$\therefore V_{i,j}^{n+1} = V_{i,j}^n + \frac{1}{L^2} [(V_{i+1,j}^n + V_{i-1,j}^n + V_{i,j+1}^n + V_{i,j-1}^n) - 4V_{i,j}^n] \delta t. \quad (2.93)$$

The “time step”  $\delta t$  is arbitrary, and so we can choose whatever we like for it. The method is simplified by setting  $\delta t = L^2/4$  (for 2D) or  $\delta t = L^2/6$  (for 3D), and so in 2D and 3D problems the iteration procedure is simply

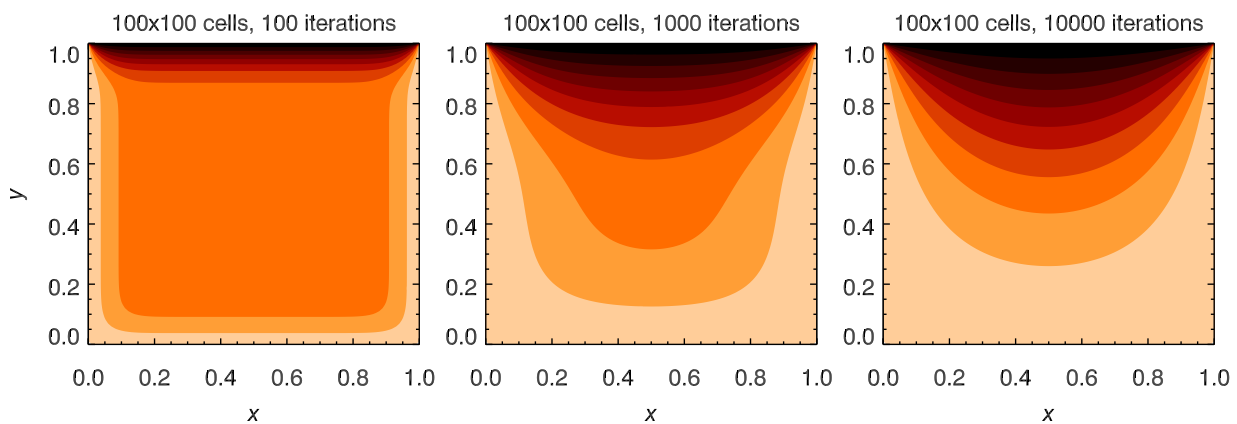
$$V_{i,j}^{n+1} = \frac{1}{4} (V_{i+1,j}^n + V_{i-1,j}^n + V_{i,j+1}^n + V_{i,j-1}^n), \quad (2.94)$$

$$V_{i,j,k}^{n+1} = \frac{1}{6} (V_{i+1,j,k}^n + V_{i-1,j,k}^n + V_{i,j+1,k}^n + V_{i,j-1,k}^n + V_{i,j,k+1}^n + V_{i,j,k-1}^n). \quad (2.95)$$

Thus, for each iteration we replace the potential on each non-boundary grid point by the average of the potentials at its nearest neighbours.

As an example, we solve the 2D problem of the potential inside a pipe of square cross section (1 m by 1 m) where the potential is specified on each of the four sides:  $V(0, y, z)=0$ ,  $V(1, y, z)=0$ ,  $V(x, 0, z)=0$  and  $V(x, 1, z)=1$  V. This is the same problem we used as an example for analytic solution (Fig. 2.7). We choose a grid spacing  $L=0.01$  m, i.e.  $100 \times 100$  cells ( $101 \times 101$  grid points), and after each iteration reset the outer grid points to the potentials on the boundary surfaces. At the two corners where  $V$  changes discontinuously from  $V=0$  to  $V=1$  V we set the potential to be the average value, i.e.  $V(0, 1, z)=V(1, 1, z)=0.5$  V. The result is contoured in Fig. 2.10 after 100, 1000, and 10,000 iterations.

Since the information on the potential can propagate from one boundary to another at a rate no faster than one cell per iteration, the number of iterations required to approach a steady-state must be much larger than the maximum number of grid points in any coordinate direction (101 in this example). In the present example we see that the number of iterations required,  $\sim 10,000$ , exceeds this by a factor of 100. Because of the large number of iterations required, very efficient codes are needed, and should use array manipulation methods available in modern programming



**Figure 2.10:** Numerical solution of Laplace's equation in 2D for the boundary condition being  $V = 0$  on all sides of the square except for the side at  $y = 1$  where  $V = 1$ . Contour levels are  $V = 0.1, 0.2, \dots, 0.9$ .

languages, rather than using do loops, when computing the average potential around each grid point, and when resetting the boundary potential. The code used in this example was written in IDL and is given in Appendix G.

## Summary of Chapter 2

### Poisson's and Laplace's equations

- Can be solved numerically, by the method of images in some cases, or analytically using Fourier methods if boundary conditions permit.

### Laplace's equation in Cartesian coordinates

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) V(x, y, z) = 0. \quad (2.96)$$

- In 2D, inside rectangular region corresponding to  $0 < x < a$  and  $0 < y < b$  with  $V(x, b) = V_0(x)$ , and  $V=0$  on the other 3 sides,

$$V(x, y) = \sum_{n=0}^{\infty} A_n \sin\left(\frac{n\pi}{a}x\right) \sinh\left(\frac{n\pi}{a}y\right). \quad (2.97)$$

- The boundary condition at  $y = b$  determines the coefficients  $A_n$  in this Fourier sine series

$$A_n = \frac{2}{a} \frac{1}{\sinh\left(\frac{n\pi}{a}b\right)} \int_0^a V_0(x) \sin\left(\frac{n\pi}{a}x\right) dx. \quad (2.98)$$

- In 3D, inside rectangular region corresponding to  $0 < x < a$ ,  $0 < y < b$  and  $0 < z < c$  with  $V(x, y, c) = V_0(x, y)$ , and  $V=0$  on the other 5 sides

$$V(x, y, z) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} A_{kl} \sinh(\gamma_{kl} z) \sin(\alpha_k x) \sin(\beta_l y), \quad (2.99)$$

$$\alpha_k \equiv \frac{k\pi}{a}, \quad \beta_l \equiv \frac{l\pi}{b} \quad \text{and} \quad \gamma_{kl}^2 \equiv \alpha_k^2 + \beta_l^2. \quad (2.100)$$

- The boundary condition at  $z = c$  determines the coefficients  $A_{kl}$  in this 2D Fourier sine series

$$A_{kl} = \frac{4}{a b \sinh(\gamma_{kl} c)} \int_0^a \int_0^b V_0(x, y) \sin(\alpha_k x) \sin(\beta_l y) dx dy. \quad (2.101)$$



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**Spherical coordinates with no dependence on  $\phi$** 

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) = 0. \quad (2.102)$$

- The general solution involves powers of  $r$  and Legendre polynomials  $P_\ell(\cos \theta)$

$$V(r, \theta) = \sum_{\ell=0}^{\infty} (A_\ell r^\ell + B_\ell r^{-(\ell+1)}) P_\ell(\cos \theta). \quad (2.103)$$

- If the space considered contains  $r = 0$  then all  $B_\ell = 0$ , or if the space considered contains  $r = \infty$  then all  $A_\ell = 0$ .
- The boundary conditions at  $r = a$  determine the coefficients  $A_\ell$  or  $B_\ell$  using orthogonality of  $P_\ell(\cos \theta)$

$$\int_{-1}^1 P_{\ell'}(\cos \theta) P_\ell(\cos \theta) d \cos \theta = \frac{2}{2\ell + 1} \delta_{\ell \ell'}. \quad (2.104)$$

**Spherical coordinates with dependence on  $\phi$** 

- The solution involves powers of  $r$  and spherical harmonics  $Y_{\ell,m}(\theta, \phi)$

$$V(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (A_{\ell,m} r^\ell + B_{\ell,m} r^{-(\ell+1)}) Y_{\ell,m}(\theta, \phi). \quad (2.105)$$

- If the space considered contains  $r = 0$  then all  $B_{\ell,m} = 0$ , or if the space considered contains  $r = \infty$  then all  $A_{\ell,m} = 0$ .
- The boundary conditions at  $r = a$  determine the coefficients  $A_{\ell,m}$  or  $B_{\ell,m}$  using orthogonality of  $Y_{\ell,m}(\theta, \phi)$

$$\oint d\Omega Y_{\ell,m}^*(\theta, \phi) Y_{\ell',m'}(\theta, \phi) = \delta_{\ell,\ell'} \delta_{m,m'}. \quad (2.106)$$

### Finite-difference method

- The potential is defined on a rectangular grid with potentials on the boundary grid points fixed by the boundary conditions.
- The potentials at the non-boundary grid points are set to reasonable values, e.g. average of boundary potential.
- The potentials on the non-boundary grid points are replaced by the average of the potentials over the nearest 6 grid points (3D) or the nearest 4 grid points (2D).
- The procedure immediately above is repeated until the potential stops changing.

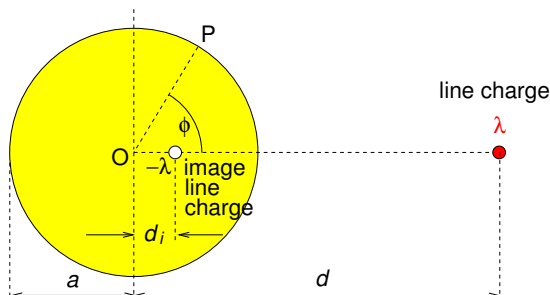
### Exercises on Chapter 2

- 2-1 Charge  $+q$  is located on the  $z$  axis a distance  $d/2$  from a grounded plane conductor in the  $x$ - $y$  plane. Find how much work was done to bring the charge to its current location using two different approaches: (a) the work done against the electrostatic force if the image charge were real and there was no grounded conductor, (b) the work done against the electrostatic force due to the induced surface charge

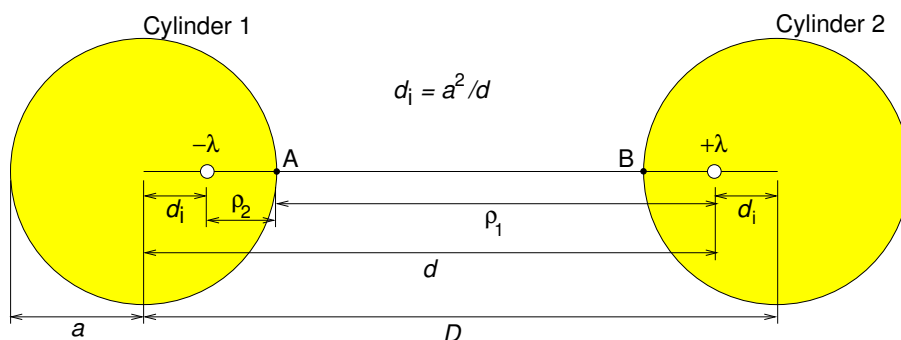
$$\sigma(x, y, 0) = \frac{-q}{2\pi} \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \quad (2.107)$$

where  $z$  is the height of the charge above the plane.

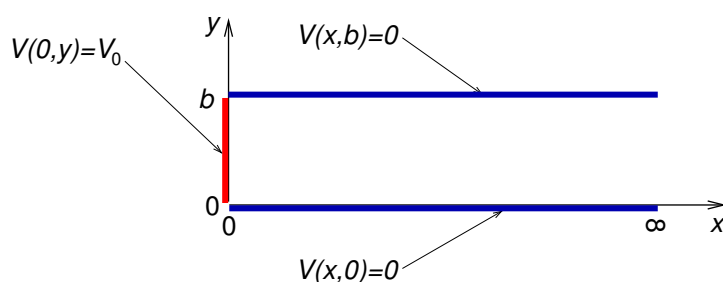
- 2-2 Charge  $+q$  is brought near to two orthogonal grounded conducting planes, one corresponding to the  $x$ - $z$  plane and the other to the  $y$ - $z$  plane. The charge is located at  $(a, b, 0)$ . Find the work done in bringing the charge from infinity to its current location (a) by using the method of images to find the potential at the location of the real charge, and (b) by considering the force on the charge as it is brought from infinity.
- 2-3 Show that the potential outside a long conducting cylinder of radius  $a$  in the presence of a long parallel line charge  $+\lambda$  at distance  $d$  is identical to the potential of the line charge and a parallel image line charge  $-\lambda$  at distance  $d_i$  from the cylinder's axis towards the real line charge (see diagram below). [Hint: draw lines to point P from the two line charges. Use the cosine rule of triangles to write the two distances in terms of  $a$ ,  $d_i$ ,  $d$  and  $\phi$  and use the formula for potential due to a line charge, and superposition, to write a formula for the potential at P. Finally require that  $V$  does not change if  $\phi$  changes.]



- 2-4 Find the capacitance of a two-wire transmission line comprising two identical parallel cylindrical conductors of radius  $a$  whose axes are separated by distance  $D$  (see diagram below). You may use the result for the potential due to a line charge near a single cylindrical conductor to find the potential difference by replacing the cylinders by equal but opposite image line charges,  $+\lambda$  and  $-\lambda$  ( $\text{C m}^{-1}$ ). The capacitance of two conductors with potential difference  $V$  and having charge  $+q$  on one and  $-q$  on the other is  $C = q/V$



- 2-5 A region of space is bounded by three plane conductors as illustrated. Find the potential everywhere between the conductors.



- 2-6 Find the potential inside the rectangular region,  $0 < x < a$ ,  $0 < y < b$  and  $0 < z < c$  with  $V(x, y, c) = V_0(x, y)$ , and  $V=0$  on the other 5 sides, where

$$V_0(x, y) = V_1 \sin\left(\frac{\pi x}{a}\right) \sin\left(\frac{3\pi y}{b}\right). \quad (2.108)$$

2-7 The potential on a non-conducting sphere of radius  $a$  is given by

$$V = V_0(3 \cos^2 \theta + \cos \theta - 1).$$

- (a) Find the potential and electric field inside the sphere.
- (b) Find the potential and electric field outside the sphere.
- (c) Find the surface charge density on the sphere as a function of  $\theta$ .

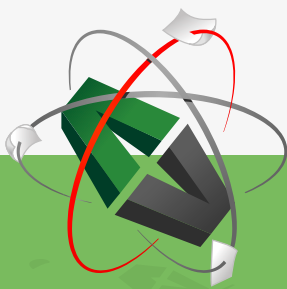
2-8 Consider a point charge on the  $z$ -axis at  $z = r'$ . Find  $V(r, \theta, \phi)$  in terms of Legendre polynomials for  $r > r'$ .

2-9 The potential on the surface of a sphere is

$$V(a, \theta, \phi) = V_1 \sin \theta \sin \phi + V_2 \sin \theta \cos \theta \sin \phi. \quad (2.109)$$

Find the potential inside the sphere.

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### 3 Multipole expansion for localised charge distribution

#### Learning objectives

- To understand how the potential for a localised charge distribution may be approximated at large distances by the first one or two non-zero terms of the multipole expansion of the potential.
- To be able to obtain the the monopole moment, dipole moment and quadrupole moment tensor in Cartesian coordinates for simple cases, and to be able approximate the potential using these.
- To be able to find the electric field and potential due to electric dipoles.
- To be able to perform a multipole expansion of the potential in spherical coordinates.

#### 3.1 Point charge on $z$ axis

Consider the potential due to a point charge  $q$  on the  $z$  axis a distance  $r'$  from the origin (Fig. 3.1a). We can obtain the expression for the potential  $V(\mathbf{r})$  for  $r > r'$  as a solution of Laplace's equation in spherical coordinates with azimuthal symmetry. For this problem we can use Coulomb's law for the potential due to point charge as the boundary condition (Exercise 2–8). The result is found to be

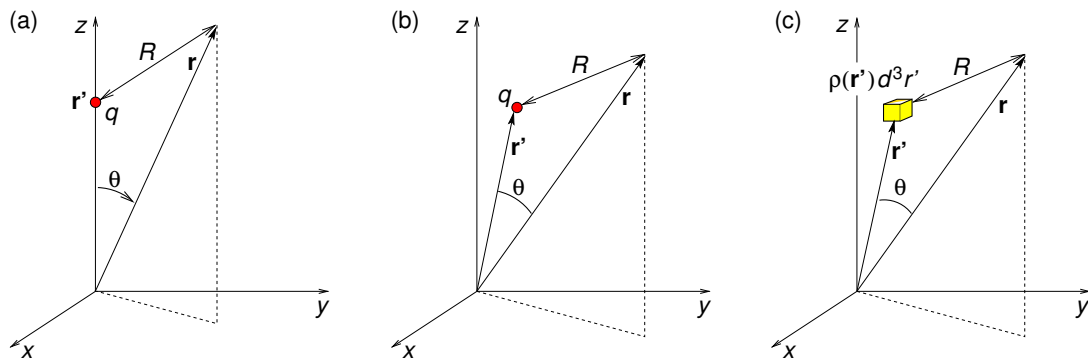
$$V(r, \theta, \phi) = \frac{q}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} (r')^{\ell} r^{-(\ell+1)} P_{\ell}(\cos \theta). \quad (3.1)$$

In Eq. 3.1,  $\theta$  is just the angle between  $\mathbf{r}$  and  $\mathbf{r}'$ , so that  $\cos \theta = \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'$  and

$$V(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} (r')^{\ell} r^{-(\ell+1)} P_{\ell}(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'). \quad (3.2)$$

Written this way the result is applies equally whether or not the charge is located on the  $z$ -axis (Fig. 3.1b).

The result given in Eq. 3.2 may be extended immediately to a localised charge distribution



**Figure 3.1:** Geometry for potential due to: (a) point charge  $q$  on  $z$  axis, (b) point charge  $q$  away from the  $z$  axis, (c) arbitrary charge density  $\rho(\mathbf{r})$ .

(Fig. 3.1c) for which we will obtain the multipole expansion of the potential. For charge density  $\rho(\mathbf{r})$ , the contribution of volume  $d^3r'$  to the potential at  $r > r'$  is

$$dV(\mathbf{r}) = \frac{\rho(\mathbf{r}')d^3r'}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} (r')^{\ell} r^{-(\ell+1)} P_{\ell}(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}'). \quad (3.3)$$

For a localised charge density  $\rho(\mathbf{r})$  the potential outside charge region is therefore

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$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} r^{-(\ell+1)} \int \rho(\mathbf{r}') (r')^{\ell} P_{\ell}(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') d^3 r'. \quad (3.4)$$

Far away from the charge region we can approximate this by first few terms

$$\begin{aligned} V(\mathbf{r}) \approx & \frac{1}{4\pi\epsilon_0 r} \int \rho(\mathbf{r}') d^3 r' + \frac{1}{4\pi\epsilon_0 r^2} \int \rho(\mathbf{r}') r' (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') d^3 r' \\ & + \frac{1}{4\pi\epsilon_0 r^3} \int \rho(\mathbf{r}') (r')^2 \frac{1}{2} [3(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')^2 - 1] d^3 r' + \dots \end{aligned} \quad (3.5)$$

The first three terms are the monopole, dipole and quadrupole terms of the *multipole expansion*, and they are proportional to  $r^{-1}$ ,  $r^{-2}$  and  $r^{-3}$ , with each having its own *multipole moment*. The lowest non-zero multipole moment is independent of the choice of coordinate origin, unlike higher multipole moments.

### 3.2 The monopole moment

The monopole potential is

$$V_{\text{mon}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r} \int \rho(\mathbf{r}') d^3 r' = \frac{q}{4\pi\epsilon_0 r} \quad (3.6)$$

where the monopole moment is just the net charge  $q$ .

### 3.3 The dipole moment

The dipole potential is

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^2} \int \rho(\mathbf{r}') r' (\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}') d^3 r' = \frac{1}{4\pi\epsilon_0 r^2} \hat{\mathbf{r}} \cdot \int \rho(\mathbf{r}') \mathbf{r}' d^3 r' = \frac{\hat{\mathbf{r}} \cdot \mathbf{p}}{4\pi\epsilon_0 r^2} \quad (3.7)$$

where the dipole moment is

$$\mathbf{p} = \int \rho(\mathbf{r}') \mathbf{r}' d^3 r', \quad (3.8)$$

or, for a collection of  $n$  point charges located at  $\mathbf{r}^{[k]}$ ,

$$\mathbf{p} = \sum_{k=1}^n q_k \mathbf{r}^{[k]}. \quad (3.9)$$

An example is the physical dipole comprising charge  $+q$  located at  $\mathbf{r}^{[1]}$  and charge  $-q$  located at  $\mathbf{r}^{[2]}$  for which  $\mathbf{p} = [(+q)\mathbf{r}^{[1]} + (-q)\mathbf{r}^{[2]}] = q\mathbf{d}$ , where  $\mathbf{d} = (\mathbf{r}^{[1]} - \mathbf{r}^{[2]})$  is the separation vector from the negative charge to the positive charge.

### 3.4 The quadrupole moment tensor

The quadrupole potential is

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3} \int \rho(\mathbf{r}') (r')^2 \frac{1}{2} [3(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')^2 - 1] d^3 r'. \quad (3.10)$$

Exercise 1–2 is to show that this can be written

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^5} \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 Q_{ij} r_i r_j \quad (3.11)$$

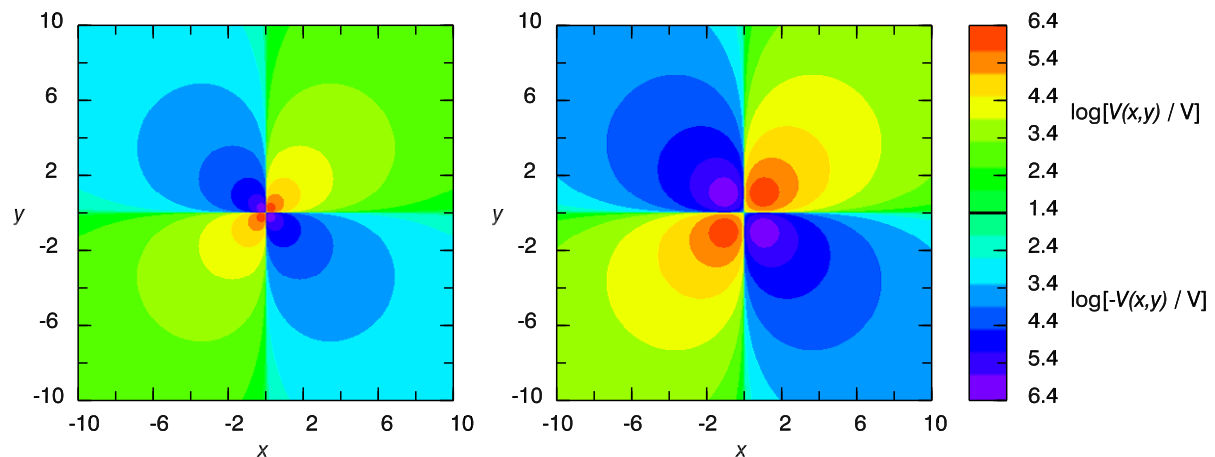
where  $r_1 \equiv x$ ,  $r_2 \equiv y$  and  $r_3 \equiv z$ , and the quadrupole moment tensor for charge density  $\rho(\mathbf{r})$ , or for  $n$  charges with the  $k$ th charge  $q_k$  located at  $\mathbf{r}^{[k]}$ , is

$$Q_{ij} = \int \rho(\mathbf{r}') [3r'_i r'_j - \delta_{ij} (r')^2] d^3 r', \quad Q_{ij} = \sum_{k=1}^n q_k [3r_i^{[k]} r_j^{[k]} - \delta_{ij} (r^{[k]})^2]. \quad (3.12)$$

As an example, consider the physical quadrupole made up of four charges at the corners of a square of side 2 m in the  $xy$  plane:  $+10^{-4}$  C at (1,1,0),  $-10^{-4}$  C at (-1,1,0),  $+10^{-4}$  C at (-1,-1,0) and  $-10^{-4}$  C at (1,-1,0), with distances in metres. The charge arrangement is depicted below together with its quadrupole moment tensor

$$\begin{array}{cc} -10^{-4} & +10^{-4} \\ +10^{-4} & -10^{-4} \end{array} \quad Q_{ij} = \begin{bmatrix} 0 & 1.2 \times 10^{-3} & 0 \\ 1.2 \times 10^{-3} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (\text{C m}^2). \quad (3.13)$$

Since the monopole and dipole terms are zero, the quadrupole moment tensor is independent of choice of origin. The potential calculated using the quadrupole moment is plotted in Fig. 3.2, where it is compared with the potential calculated for the four charges directly using Coulomb's law. Another example of the quadrupole moment tensor is given in Exercise 3–4.



**Figure 3.2:** Potential due to the four charges and their quadrupole moment tensor (Eq. 3.13) as described above: left — calculated using the quadrupole moment; right — calculated for the four charges directly from Coulomb's law. (Potentials higher than  $10^{6.4}$  V are set to  $10^{6.4}$  V).



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### 3.5 Electric dipoles

Electric dipoles are important because, as we shall see in the next chapter, when dielectrics are subject to an electric field they acquire an electric dipole moment. Also, oscillating dipoles are efficient radiators of electromagnetic waves, and are discussed in the 2nd book in this series “Essential Electrodynamics”.

For a dipole  $\mathbf{p}$  at the origin pointing in the  $z$ -direction

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{p \cos \theta}{r^2}. \quad (3.14)$$

The components of the electric field,  $\mathbf{E}_{\text{dip}}(\mathbf{r}) = -\nabla V_{\text{dip}}(\mathbf{r})$ , are

$$E_r = -\frac{\partial V}{\partial r} = \frac{2p \cos \theta}{4\pi\epsilon_0 r^3}, \quad E_\theta = -\frac{1}{r} \frac{\partial V}{\partial \theta} = \frac{p \sin \theta}{4\pi\epsilon_0 r^3}, \quad E_\phi = -\frac{1}{r \sin \theta} \frac{\partial V}{\partial \phi} = 0.$$

$$\therefore \mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3} \left[ 2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}} \right]. \quad (3.15)$$

For a dipole at the origin pointing in an arbitrary direction

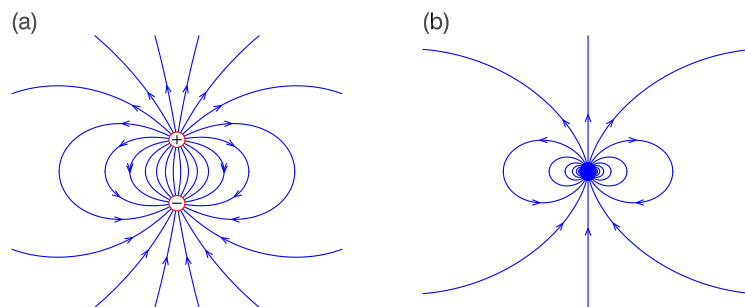
$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = -\nabla \left( \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{r^3} \right), \quad (3.16)$$

$$= -\frac{1}{4\pi\epsilon_0} \left[ \mathbf{p} \cdot \mathbf{r} \nabla \left( \frac{1}{r^3} \right) + \frac{1}{r^3} \nabla (\mathbf{p} \cdot \mathbf{r}) \right], \quad (3.17)$$

$$= -\frac{1}{4\pi\epsilon_0} \left[ \mathbf{p} \cdot \mathbf{r} (-3) \left( \frac{1}{r^4} \right) \hat{\mathbf{r}} + \frac{1}{r^3} \nabla (p_x x + p_y y + p_z z) \right], \quad (3.18)$$

$$\mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3} [3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]. \quad (3.19)$$

While a physical dipole comprises two charges,  $+q$  and  $-q$ , separated by distance  $d$  and has dipole moment  $\mathbf{p} = q\mathbf{d}$  where  $\mathbf{d}$  extends from  $-q$  to  $+q$ , a pure dipole can be thought of as a physical dipole with dipole moment  $\mathbf{p} = q\mathbf{d}$ , with  $q = p/d$ , in the limit that  $d \rightarrow 0$ . Field lines of a physical dipole and a pure dipole are shown in Fig. 3.3. If we were to zoom out so that the scale was very much larger than the separation between the two point charges in part (a) the two fields would then appear to be identical.



**Figure 3.3:** Field lines of (a) a physical dipole, and (b) a pure dipole.

### 3.6 Multipole expansion in spherical coordinates

In addition to the multipole expansion in Cartesian coordinates, one can perform a multipole expansion of a localised charge distribution in spherical coordinates. We give this without proof,

$$V(r, \theta, \phi) = \frac{1}{\varepsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{q_{\ell,m}}{2\ell+1} r^{-(\ell+1)} Y_{\ell,m}(\theta, \phi), \quad (3.20)$$

where the spherical multipole moments are

$$q_{\ell,m} = \int Y_{\ell,m}^*(\theta', \phi') r'^{\ell} \rho(\mathbf{r}') d^3 r'. \quad (3.21)$$

These are directly related to the Cartesian multipole moments, e.g.

$$q_{00} = \frac{1}{4\pi} q, \quad q_{11} = -\sqrt{\frac{3}{8\pi}} (p_x - ip_y), \quad q_{10} = \sqrt{\frac{3}{4\pi}} p_z. \quad (3.22)$$

## Summary of Chapter 3

### Multipole expansion for a localised charge distribution

- Far from a localised charge distribution the potential is

$$V(\mathbf{r}) \approx \frac{q}{4\pi\epsilon_0 r} + \frac{\hat{\mathbf{r}} \cdot \mathbf{p}}{4\pi\epsilon_0 r^2} + \frac{1}{4\pi\epsilon_0 r^5} \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 Q_{ij} r_i r_j + \dots \quad (3.23)$$

- Dipole moment for charge density  $\rho(\mathbf{r})$ , and point charges

$$\mathbf{p} = \int \rho(\mathbf{r}') \mathbf{r}' d^3 r', \quad \mathbf{p} = \sum_{i=1}^n q_i \mathbf{r}^{[i]} \quad (3.24)$$

- Quadrupole moment tensor for charge density  $\rho(\mathbf{r})$  and point charges,

$$Q_{ij} = \int \rho(\mathbf{r}') [3r'_i r'_j - \delta_{ij} (r')^2] d^3 r', \quad Q_{ij} = \sum_{k=1}^n q_k [3r_i^{[k]} r_j^{[k]} - \delta_{ij} (r^{[k]})^2] \quad (3.25)$$

- Potential and electric field of a dipole  $\mathbf{p}$  at the origin pointing in the  $z$ -direction

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}, \quad \mathbf{E}_{\text{dip}}(\mathbf{r}) = \frac{p}{4\pi\epsilon_0 r^3} [2 \cos \theta \hat{\mathbf{r}} + \sin \theta \hat{\boldsymbol{\theta}}] \quad (3.26)$$

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## Exercises on Chapter 3

3–1 On the surface of a non-conducting sphere of radius  $a$  is surface charge density  $\sigma(a, \theta, \phi) = \sigma_0 \cos^3 \theta$ . Find the dipole moment of the sphere.

3–2 The quadrupole potential is

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^3} \int \rho(\mathbf{r}') (r')^2 \frac{1}{2} [3(\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}')^2 - 1] d^3 r'. \quad (3.27)$$

Show that it can be written as

$$V_{\text{quad}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r^5} \frac{1}{2} \sum_{i=1}^3 \sum_{j=1}^3 Q_{ij} r_i r_j \quad (3.28)$$

where the quadrupole moment tensor is

$$Q_{ij} = \int \rho(\mathbf{r}') [3r'_i r'_j - \delta_{ij} (r')^2] d^3 r'. \quad (3.29)$$

[This exercise is easy using index notation.]

3–3 A physical quadrupole is made up of four charges lined up along the  $z$  axis:  $-q_0$  at  $(0,0,-2a)$ ,  $+q_0$  at  $(0,0,-a)$ ,  $+q_0$  at  $(0,0,a)$  and  $-q_0$  at  $(0,0,2a)$ . (a) Obtain the quadrupole moment. (b) Find the potential at  $\mathbf{r} = (b, b, 0)$  for  $b \gg a$ .

3–4 Charge  $-q$  is located at the origin and charge  $+q$  is located at

$$(x, y, z) = (a \sin \theta_0 \cos \phi_0, a \sin \theta_0 \sin \phi_0, a \cos \theta_0).$$

(a) Find the non-zero moments of the multipole expansion of the potential in Cartesian coordinates, i.e.  $q$ ,  $\mathbf{p}$ ,  $Q_{ij}$  (if non-zero), and use these moments in the multipole expansion in Cartesian coordinates to find the potential at  $(x, y, z) = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$  where  $r \gg a$ . (b) Find the non-zero moments of the multipole expansion of the potential in spherical coordinates, i.e.

$$q_{\ell m} = \int Y_{\ell m}^*(\theta', \phi') r'^{\ell} \rho(\mathbf{r}') d^3 r', \quad (3.30)$$

and use these moments in the multipole expansion in spherical coordinates to find the potential at  $(r, \theta, \phi)$  where  $r \gg a$ . Compare the result with that from part (a).

## 4 Macroscopic and microscopic dielectric theory

### Learning objectives

- To understand that dielectric materials when placed in an electric field acquire an electric dipole moment per unit volume described by the polarisation field, and that this leads to polarisation volume and surface charge densities.
- To learn how Gauss' law must be modified to include polarisation charges, and that it may be written in terms of a new field called the electric displacement.
- To learn that for many dielectrics the polarisation is proportional to the electric field with the constant of proportionality being the electric susceptibility, and to learn how the susceptibility is related to the permittivity.
- To be able to calculate the capacitance of capacitors having a simple geometry.
- To know and be able to apply the boundary conditions on the potential, electric field and displacement field at an interface between two dielectrics.
- To understand, on a microscopic level, the causes of polarisation, including the orientational polarisability of polar molecules, electronic polarisability of non-polar molecules and the ionic polarisability of crystals.
- To know that the electric field at the atomic scale responsible for aligning or stretching atomic or molecular dipoles is different to the macroscopic electric field in the dielectric, and that the Clausius-Mossotti formula accounts for this in relating the molecular polarisability to the permittivity.

### 4.1 Dielectrics

Electrons in *dielectrics* or *insulators* are attached to specific atoms or molecules. This is very different to in conductors where some fraction of electrons are free to move throughout the material. Some molecules have permanent electric dipoles. If an electric field is applied to a dielectric, molecules rotate or atoms become stretched, and the material becomes *polarised*. A polarised material has a net dipole moment per unit volume called the *polarisation field*,  $\mathbf{P}(\mathbf{r})$  (units:  $\text{C m}^{-2}$ ).

## 4.2 Macroscopic dielectric theory

By working out the potential of a polarised object we shall identify parts due to volume and surface polarisation charges. The electrostatic potential due to a point dipole located at  $\mathbf{r}'$  is

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \hat{\mathbf{R}}}{R^2} \quad (4.1)$$

where  $\mathbf{R} = (\mathbf{r} - \mathbf{r}')$  as usual. For a polarised dielectric the dipole moment due to volume element  $d^3r'$  at position  $\mathbf{r}'$  is  $\mathbf{P}(\mathbf{r}')d^3r'$ , and so for the whole object of volume  $\mathcal{V}$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\mathbf{P}(\mathbf{r}') \cdot \hat{\mathbf{R}}}{R^2} d^3r' = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \mathbf{P}(\mathbf{r}') \cdot \left( \frac{\hat{\mathbf{R}}}{R^2} \right) d^3r'. \quad (4.2)$$

But we know (Eq. B.9) that  $R^{-2}\hat{\mathbf{R}} = \nabla' R^{-1}$  where  $\nabla' = \left( \hat{\mathbf{x}} \frac{\partial}{\partial x'} + \hat{\mathbf{y}} \frac{\partial}{\partial y'} + \hat{\mathbf{z}} \frac{\partial}{\partial z'} \right)$ , and so

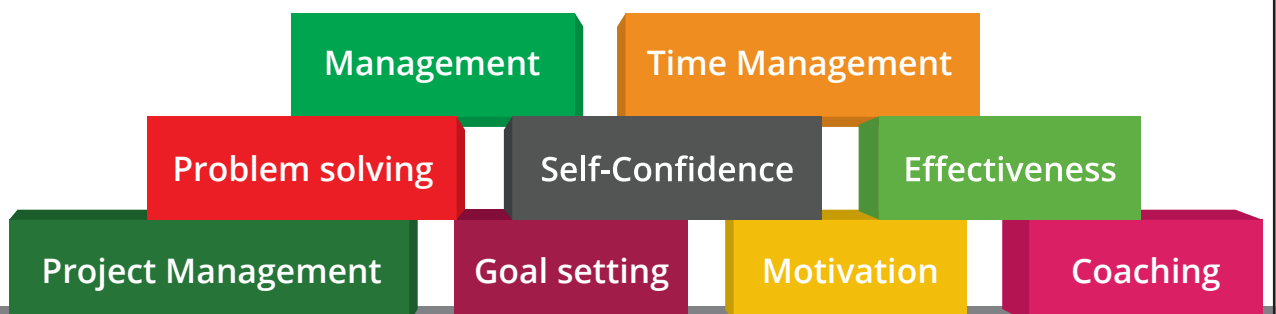
$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \mathbf{P}(\mathbf{r}') \cdot \nabla' \left( \frac{1}{R} \right) d^3r'. \quad (4.3)$$

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Then we can use the product rule  $\nabla' \cdot (u\mathbf{A}) = (\nabla' u) \cdot \mathbf{A} + u(\nabla' \cdot \mathbf{A})$  to integrate by parts

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \nabla' \cdot \left[ \frac{\mathbf{P}(\mathbf{r}')}{R} \right] d^3r' - \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{R} d^3r', \quad (4.4)$$

and then use Gauss' theorem on the 1st integral to obtain

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_S \frac{\mathbf{P}(\mathbf{r}') \cdot \hat{\mathbf{n}}}{R} dS' - \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\nabla' \cdot \mathbf{P}(\mathbf{r}')}{R} d^3r', \quad (4.5)$$

where surface  $S$  bounds volume  $\mathcal{V}$ .

Comparing Eq. 4.5 with the potential due to surface and volume charge densities (Eqs. 1.20) we see that

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint_S \frac{\sigma_{\text{pol}}(\mathbf{r}')}{R} dS' + \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{\rho_{\text{pol}}(\mathbf{r}')}{R} d^3r', \quad (4.6)$$

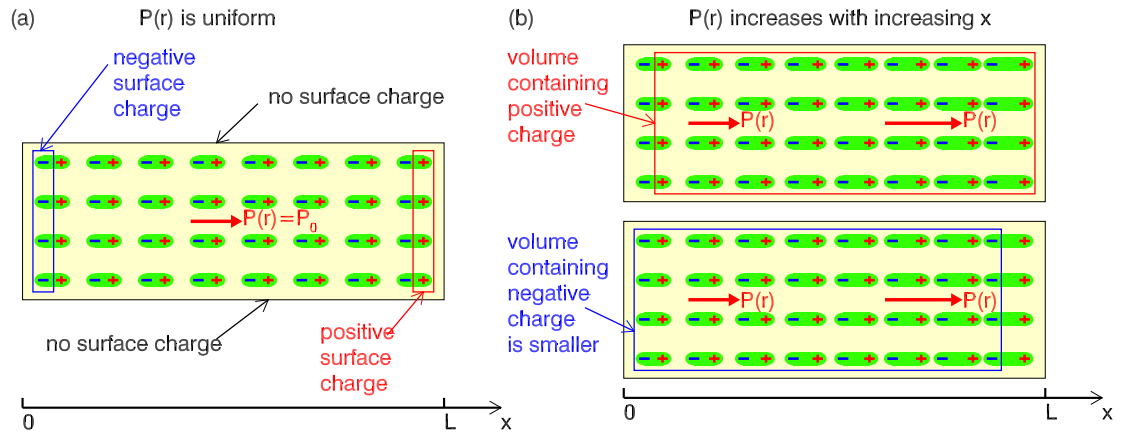
and the “polarisation” or “bound” volume and surface charge densities are

$$\rho_{\text{pol}}(\mathbf{r}) = -\nabla \cdot \mathbf{P}(\mathbf{r}) \quad (\text{unit: C m}^{-3}), \quad (4.7)$$

$$\sigma_{\text{pol}}(\mathbf{r}) = \mathbf{P}(\mathbf{r}) \cdot \hat{\mathbf{n}} \quad (\text{unit: C m}^{-2}). \quad (4.8)$$

Figure 4.1 is a cartoon which attempts to illustrate how surface and volume polarisation charges occur at an atomic level in a dielectric material. In Fig. 4.1(a) we show a microscopic rectangular slab of dielectric of length  $L$  containing an array of atoms which have been stretched to form tiny dipoles in the presence of an applied electric field. The material is uniformly polarised, with  $\mathbf{P}(\mathbf{r}) = P_0 \hat{\mathbf{x}}$ . On the right hand side of the dielectric  $\sigma_{\text{pol}}(L, y, z) = \mathbf{P} \cdot \hat{\mathbf{x}} = +P_0$  (positive surface charge), whereas on the left hand side  $\sigma_{\text{pol}}(0, y, z) = \mathbf{P} \cdot (-\hat{\mathbf{x}}) = -P_0$  (negative surface charge). There is no surface polarisation charge present on the top and bottom surfaces as  $\mathbf{P} \cdot \hat{\mathbf{n}} = 0$  there. Also, because  $\mathbf{P}$  is uniform,  $\rho_{\text{pol}}(\mathbf{r}) = -\nabla \cdot \mathbf{P}(\mathbf{r}) = 0$  and so there is no volume polarisation charge either.

In Fig. 4.1(b) the dielectric has been polarised non-uniformly and in the rectangular slab shown  $\mathbf{P}(\mathbf{r}) = (P_0 + ax) \hat{\mathbf{x}}$ . There is still negative surface charge on the left hand side and positive surface charge on the right hand side, but they have unequal magnitudes:  $\sigma_{\text{pol}}(0, y, z) = -P_0$  and  $\sigma_{\text{pol}}(L, y, z) = (P_0 + aL)$ . Inside the slab  $\nabla \cdot \mathbf{P}(\mathbf{r}) = a$  and so the polarisation volume charge density is negative,  $\rho_{\text{pol}}(\mathbf{r}) = -a$ . This can be expected as, although there are equal



**Figure 4.1:** Explanation of polarisation charge: (a) uniform polarisation giving rise to surface polarisation charge on the surfaces defined by  $x = 0$  and  $x = L$ , (b) non-uniform polarisation giving rise also to a volume polarisation charge density. The displacement of electron distributions from the nuclei are exaggerated.

numbers of real positive and negative charges, the negative charges occupy a smaller volume than the positive charges (compare the two boxes in Fig. 4.1b) giving rise to a net volume polarisation charge density which is negative. If the slab has cross-sectional area  $S$ , then the total polarisation charge is  $q_{\text{pol}} = S[(-P_0) + (P_0 + aL) + (-a \times L)] = 0$ .

### 4.3 Gauss' law and the electric displacement

Inside dielectrics, Gauss' Law must include both free charge  $\rho_f$  and polarisation charge  $\rho_{\text{pol}}$

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho_f(\mathbf{r}) + \rho_{\text{pol}}(\mathbf{r})}{\varepsilon_0} = \frac{\rho_f(\mathbf{r}) - \nabla \cdot \mathbf{P}(\mathbf{r})}{\varepsilon_0}. \quad (4.9)$$

$$\therefore \nabla \cdot [\varepsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})] = \rho_f(\mathbf{r}). \quad (4.10)$$

Hence, Gauss' law in matter becomes

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho_f(\mathbf{r}), \quad \oint \mathbf{D} \cdot d\mathbf{S} = Q_f, \quad (4.11)$$

where  $\mathbf{D}(\mathbf{r}) = [\varepsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})]$  is the *displacement field* (unit:  $\text{C m}^{-2}$ ).

### 4.3.1 Energy density

We shall now find the energy density of an electric field in a dielectric, starting with

$$W = \frac{1}{2} \int_{\text{all space}} \rho_f(\mathbf{r}) V(\mathbf{r}) d^3r, \quad (4.12)$$

and using Gauss' law to replace  $\rho_f$

$$W = \frac{1}{2} \int_{\text{all space}} (\nabla \cdot \mathbf{D}) V d^3r, \quad (4.13)$$

$$= \frac{1}{2} \int_{\text{all space}} \nabla \cdot (V\mathbf{D}) d^3r - \frac{1}{2} \int_{\text{all space}} (\nabla V) \cdot \mathbf{D} d^3r, \quad (4.14)$$

where we have used a product rule to integrate by parts. Then using Gauss' theorem on the 1st integral and  $\mathbf{E} = -\nabla V$  on the 2nd integral we find

$$W = \frac{1}{2} \oint_{S \text{ at } \infty} (V\mathbf{D}) \cdot d\mathbf{S} + \frac{1}{2} \int_{\text{all space}} \mathbf{E} \cdot \mathbf{D} d^3r, \quad (4.15)$$



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$$\therefore W = \frac{1}{2} \int_{\text{all space}} \mathbf{E} \cdot \mathbf{D} d^3r, \quad (4.16)$$

since the surface integral is zero because for a localised charge distribution  $VD \sim r^{-3}$  and  $S \sim r^2$  as  $r \rightarrow \infty$ . Hence the energy density is  $\frac{1}{2} \mathbf{E} \cdot \mathbf{D}$ .

### 4.3.2 Linear dielectrics

For linear dielectrics, i.e. most materials unless the electric field is too high,

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E} \quad (4.17)$$

where  $\chi_e$  is the *electric susceptibility*. Then

$$\mathbf{D} = (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \varepsilon_0(1 + \chi_e) \mathbf{E} = \varepsilon \mathbf{E} = \varepsilon_r \varepsilon_0 \mathbf{E} = K \varepsilon_0 \mathbf{E} \quad (4.18)$$

$\varepsilon$  is the *permittivity* and  $\varepsilon_r = K$  is its *relative permittivity* or *dielectric constant*. Some values of relative permittivity are:  $\varepsilon_r = 1$  (vacuum),  $\varepsilon_r = 1.00055$  (Nitrogen),  $\varepsilon_r = 5.9$  (salt),  $\varepsilon_r = 80.1$  (water).

### 4.3.3 Capacitors

A capacitor is a device for storing charge, typically comprising two conductors separated from each other by air or a dielectric, the simplest being the parallel plate capacitor comprising two parallel conducting plates separated by a small gap. The potential difference between two conductors, “a” and “b”, say, is

$$V = (V_a - V_b) = - \int_b^a \mathbf{E} \cdot d\mathbf{r}. \quad (4.19)$$

The electric field due to the two charged conductors which have surface areas  $S_a$  and  $S_b$  is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_{S_b+S_a} \frac{\sigma(\mathbf{r}')}{R^2} \hat{\mathbf{R}} dS' \quad (4.20)$$

where  $\sigma(\mathbf{r})$  is the surface charge density on the conductors. If the charge on each conductor is doubled,  $\mathbf{E}$  is doubled and  $V$  must also be doubled, and so the charge stored  $q$  is proportional to  $V$ . The constant of proportionality is the *capacitance*  $C$ , such that  $q = CV$ . The unit of capacitance is the farad, and is named after Michael Faraday.

To charge a capacitor, you must transport electrons from the positive plate to the negative plate. If the capacitor has charge  $q'$ , the potential difference is  $V = q'/C$ , and so the work done to add a small additional charge  $dq'$  to a capacitor which already stores charge  $q'$  is

$$dW = V dq' = \frac{q'}{C} dq'. \quad (4.21)$$

We can integrate this from  $q' = 0$  to  $q' = q$  to determine the work done to charge an initially uncharged capacitor with charge  $q$

$$W = \int_0^q \frac{q'}{C} dq', \quad \therefore W = \frac{1}{2} \frac{q^2}{C} = \frac{1}{2} CV^2. \quad (4.22)$$

A schematic diagram of a parallel plate capacitor is shown in Fig. 4.2(a) where the plate separation is greatly exaggerated. The capacitance can be calculated as follows: (i) neglecting fringing fields (bulging out around the edges of the plates) the electric field is located solely between the plates and is perpendicular to the plates, (ii) using Gauss' law the surface charge density is  $\sigma_0 = \varepsilon_0 E$ , (iii) the total charge is

$$q = S\sigma_0 = S\varepsilon_0 E = S\varepsilon_0 \frac{V}{a}. \quad \therefore C = \frac{S\varepsilon_0}{a}. \quad (4.23)$$

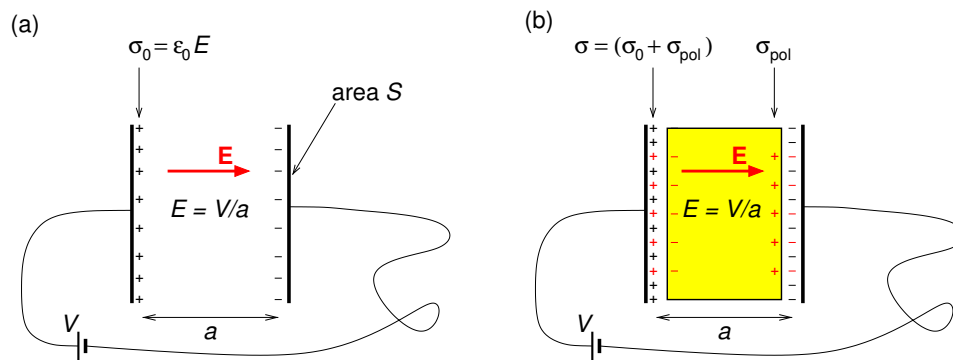
If a dielectric is placed between the plates,  $E$  is unchanged (it depends only on  $V$  and  $a$ ), but the dielectric becomes polarised  $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$  and acquires surface polarisation charge (see Fig. 4.2b)

$$\sigma_{\text{pol}} = \mathbf{P} \cdot \hat{\mathbf{n}} = \chi_e \varepsilon_0 \mathbf{E} \cdot \hat{\mathbf{n}}. \quad (4.24)$$

To keep  $\mathbf{E}$  the same, more charge (shown in red) must flow onto the plates to cancel the polarisation charge such that

$$\sigma = (\sigma_0 + \sigma_{\text{pol}}) = (\varepsilon_0 E + \varepsilon_0 \chi_e E) = (1 + \chi_e) \varepsilon_0 E = \varepsilon E. \quad \therefore C = \frac{S\varepsilon}{a}. \quad (4.25)$$





**Figure 4.2:** Parallel plate capacitor filled with: (a) air ( $\epsilon \approx \epsilon_0$ ), (b) dielectric with permittivity  $\epsilon$ . The separation between the plates is greatly exaggerated, and the gap between the dielectric and the plates is negligible.

Practical capacitors used in ordinary circuits are made of two long (to make  $S$  large) strips of metal foil separated by a thin (to make  $a$  small) insulating layer, and rolled up into a cylinder (to save space).

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#### 4.4 Boundary conditions on $V$ , $\mathbf{E}$ and $\mathbf{D}$

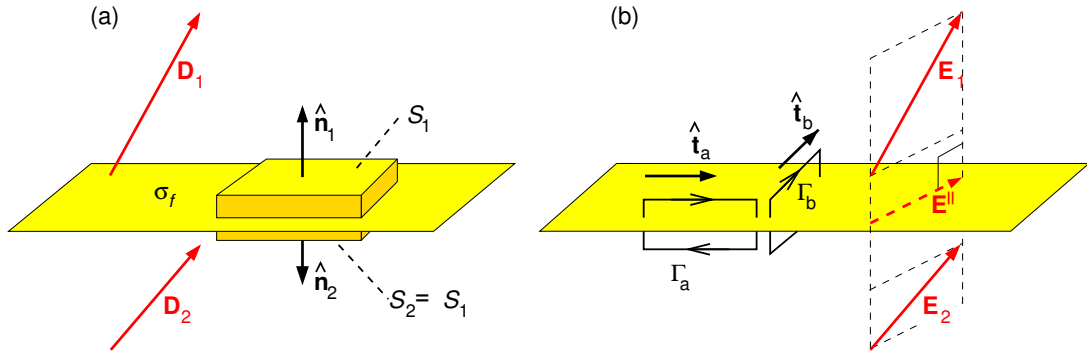
The boundary condition on  $V$  at the interface between two materials is simply that  $V$  must be continuous across the interface; otherwise at the interface the electric field would be infinite there.

To obtain the boundary conditions on  $\mathbf{E}$  and  $\mathbf{D}$  at the interface between two materials we split up the problem into finding the conditions on the components normal to and parallel to the interface. We obtain the boundary conditions on the normal components by applying Gauss' law to an infinitesimally-thin Gaussian pill-box spanning the interface between two materials as in Fig. 4.3(a). Since both the surface polarisation charge and the free surface charge will enter into Gauss' law for  $\mathbf{E}$ , whereas only the free surface charge enters into Gauss' law for  $\mathbf{D}$ , the boundary condition on the normal component of  $\mathbf{D}$  is much more useful. Then,

$$\oint_{S_1+S_2} \mathbf{D}(\mathbf{r}) \cdot \hat{\mathbf{n}} dS = Q_{f, \text{encl}}, \quad \mathbf{D}_1 \cdot \hat{\mathbf{n}}_1 S_1 + \mathbf{D}_2 \cdot \hat{\mathbf{n}}_2 S_2 = S_1 \sigma_f. \quad (4.26)$$

$$\therefore (\mathbf{D}_1 - \mathbf{D}_2) \cdot \hat{\mathbf{n}}_1 = \sigma_f, \quad (4.27)$$

since  $S_2 = S_1$ , and so the normal component of  $\mathbf{D}$  is unchanged across the interface between two materials, unless there is free surface charge on the boundary.



**Figure 4.3:** (a) Gaussian pill box used to find the boundary condition on the normal component of  $\mathbf{D}$ ; (b) integration loops used to find the boundary condition on the parallel component of  $\mathbf{E}$ .

In electrostatics the electric field is conservative, and so

$$\oint_{\Gamma} \mathbf{E}(\mathbf{r}) \cdot d\mathbf{r} = 0 \quad (4.28)$$

for any “loop”. Using infinitesimally narrow loops  $\Gamma_a$  or  $\Gamma_b$  of lengths  $\ell_a$  and  $\ell_b$  spanning the

interface, and with  $\hat{\mathbf{t}}_a$  orthogonal to  $\hat{\mathbf{t}}_b$ , as in Fig. 4.3(b), we obtain

$$(\mathbf{E}_1 - \mathbf{E}_2) \cdot (\ell_a \hat{\mathbf{t}}_a) = 0, \quad \text{and} \quad (\mathbf{E}_1 - \mathbf{E}_2) \cdot (\ell_b \hat{\mathbf{t}}_b) = 0. \quad (4.29)$$

Adding the components of  $\mathbf{E}$  parallel to  $\hat{\mathbf{t}}_a$  and to  $\hat{\mathbf{t}}_b$  we have the component of the electric field parallel to the interface,  $\mathbf{E}^{\parallel}$ , and this must be the same on either side of the boundary,

$$\mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel}. \quad (4.30)$$

## 4.5 Microscopic dielectric theory

In the presence of an electric field, atoms become stretched or molecules rotate thereby acquiring on average a dipole moment  $\mathbf{p} = \alpha_{\text{pol}} \mathbf{E}$  where  $\alpha_{\text{pol}}$  is the *polarisability*.

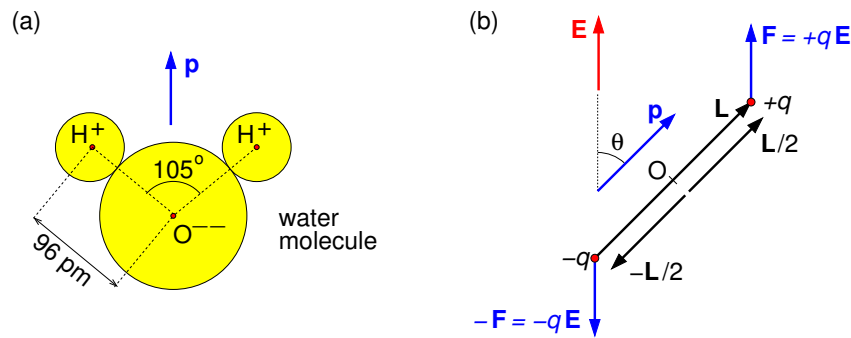
### 4.5.1 Orientational polarisability of polar molecules

Molecules having a permanent electric dipole are called *polar molecules*. A common example is the water molecule illustrated in Fig. 4.4(a), but there are many other examples including carbon monoxide and ozone. In the presence of a uniform electric field the net force on an electric dipole is zero, but a permanent electric dipole such as a water molecule will experience a torque tending to align its dipole moment with the electric field (Fig. 4.4b),

$$\begin{aligned} \mathbf{N} &= \left( \frac{\mathbf{L}}{2} \right) \times (q\mathbf{E}) + \left( -\frac{\mathbf{L}}{2} \right) \times (-q\mathbf{E}) = q\mathbf{L} \times \mathbf{E}, \\ \therefore \mathbf{N} &= \mathbf{p} \times \mathbf{E}. \end{aligned} \quad (4.31)$$

At a finite temperature, collisions with other molecules regularly mis-align individual molecules such that, on average, only partial alignment is maintained.

The lowest energy state of a polar molecule in an electric field corresponds to when  $\mathbf{p}$  is aligned parallel to  $\mathbf{E}$ , and the work done to rotate  $\mathbf{p}$  so that it is oriented at angle  $\theta$  to the electric field



**Figure 4.4:** (a) Diagram of water molecule showing its dipole moment. (b) Forces on individual charges of a physical dipole.

is

$$W(\theta) = \int_0^\theta N(\theta') d\theta' = \int_0^\theta pE \sin \theta' d\theta' = pE(1 - \cos \theta). \quad (4.32)$$

At finite temperatures, the probability density of  $W$ , and consequently of  $\cos \theta$ , can be obtained using the Boltzmann distribution, i.e.

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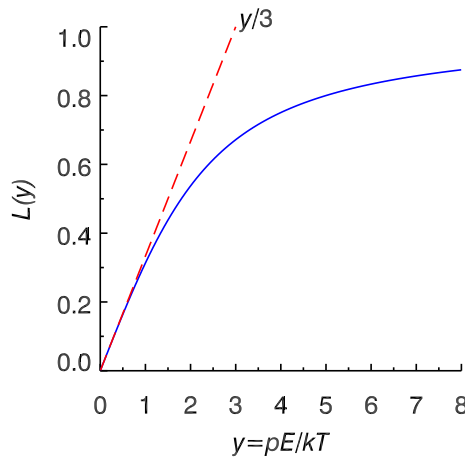
$$f(W) \propto e^{-W(\theta)/kT} \propto e^{pE \cos \theta / kT}, \quad \therefore f(\cos \theta) \propto e^{y \cos \theta}, \quad (4.33)$$

where  $y \equiv pE/kT$ . For this probability density, the time-average of a dipole moment's component parallel to  $\mathbf{E}$  is

$$\langle p \cos \theta \rangle = p \frac{\int_{-1}^1 \cos \theta e^{y \cos \theta} d \cos \theta}{\int_{-1}^1 e^{y \cos \theta} d \cos \theta} = p \left[ \coth(y) - \frac{1}{y} \right] \equiv p L(y), \quad (4.34)$$

where  $L(y)$  is the “Langevin function” which is plotted in Fig. 4.5. For ordinary electric fields and temperatures,  $y \ll 1$  such that  $L(y) \approx y/3$ , and the orientational polarisability obeys the “Curie law” named after Pierre Curie (1859–1906) who discovered the  $1/T$  dependence,

$$\alpha_{\text{pol}} = \frac{\langle \mathbf{p} \cdot \hat{\mathbf{E}} \rangle}{E} \approx \frac{p^2}{3kT}. \quad (4.35)$$



**Figure 4.5:** The Langevin function (Eq. 4.34).

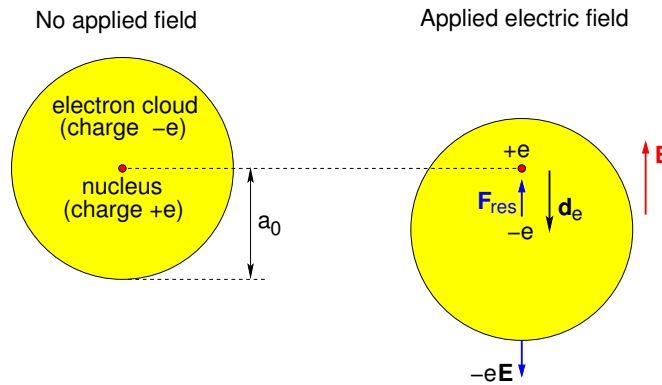
#### 4.5.2 Electronic polarisability of non-polar molecules

In an attempt to understand how atoms acquire an electric dipole moment when placed in an electric field we approximate an atom by a nucleus screened by its inner electrons (effective charge  $+e$ ) and a spherical valence electron “cloud” (charge  $-e$ ) of radius  $a_0 \approx 10^{-10}$  m (Fig. 4.6).

In the presence of an applied electric field, as the electron cloud becomes displaced with respect to the nucleus by an amount  $\mathbf{d}_e$  it experiences a restoring force

$$\mathbf{F}_{\text{res}} = -\frac{(d_e/a_0)^3 e^2}{4\pi\epsilon_0 d_e^2} \hat{\mathbf{d}}_e = -\frac{e^2}{4\pi\epsilon_0 a_0^3} \mathbf{d}_e. \quad (4.36)$$

This is equivalent to Hooke's law with "spring constant"  $\kappa \approx e^2/4\pi\epsilon_0 a_0^3$ .



**Figure 4.6:** Semi-classical model of an atom without, and with, an applied electric field (displacement greatly exaggerated).

The net force on the valence electron is then

$$m_e \frac{d^2 \mathbf{d}_e}{dt^2} = -e\mathbf{E} - \kappa \mathbf{d}_e, \quad (4.37)$$

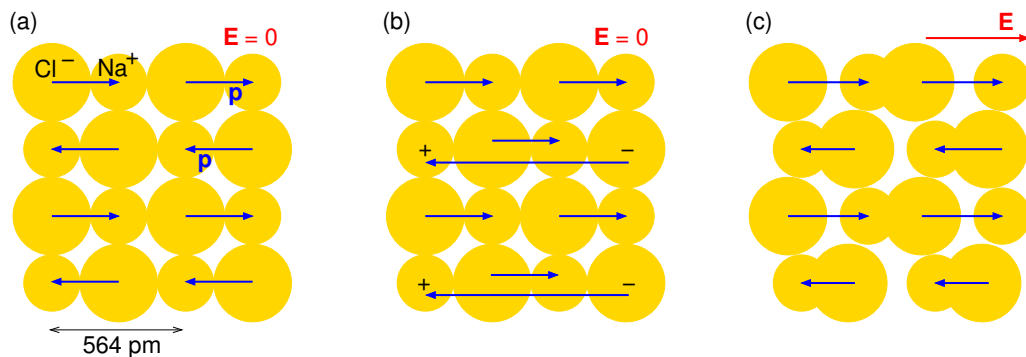
which must be zero in electrostatics. Hence there will be an equilibrium displacement, dipole moment and molecular polarisability

$$\mathbf{d}_e = -e \frac{\mathbf{E}}{\kappa}, \quad \mathbf{p} = -e \mathbf{d}_e = e^2 \frac{\mathbf{E}}{\kappa}, \quad \alpha_{\text{pol}} = \frac{p}{E} = \frac{e^2}{\kappa} = 4\pi\epsilon_0 a_0^3. \quad (4.38)$$

### 4.5.3 Ionic polarisability of crystals

Consider a tiny piece of a salt crystal sketched in Fig. 4.7(a) comprising  $\text{Na}^+$  and  $\text{Cl}^-$  ions in a face-centred cubic structure. In the absence of an applied electric field the crystal has zero net dipole moment. Within the crystal, physical dipoles are associated with pairs of  $\text{Na}^+$  and  $\text{Cl}^-$  ions but they cancel out over the volume of the crystal as shown (there are equal numbers of dipoles pointing to the left as there are to the right). This might look a bit suspicious, as you could draw all the dipoles to point to the right as in Fig. 4.7(b). However, for this rather

artificial construct there will remain unpaired ions, resulting in negative surface charge on the right and positive surface charge on the left, and the dipoles associated with the surface charges would cancel the internal dipoles.



**Figure 4.7:** Face-centred cubic crystal structure of NaCl (salt) showing: (a) locations of  $\text{Na}^+$  and  $\text{Cl}^-$  ions and how dipoles associated with ion pairs will cancel over the volume, (b) an alternative way of showing the cancellation as in this case internal dipoles are cancelled by those associated with surface charge, (c) a net dipole moment if an electric field is applied (ion displacements exaggerated).

Fig. 4.7(c) shows negative ions displaced to the left and positive ions displaced to the right in response to an applied electric field such that the internal dipoles no longer cancel, but give

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rise to an overall polarisation in the direction of the applied field. Usually for crystals (NaCl is an exception) the direction of  $\mathbf{E}$  relative to crystal planes matters, and  $\mathbf{p}$  is related to  $\mathbf{E}$  by the polarisability tensor, rather than the (scalar) polarisability.

#### 4.5.4 Clausius-Mossotti formula

In an amorphous substance made of non-polar molecules, or an isotropic crystal such as NaCl, placed in an external electric field  $\mathbf{E}_{\text{ext}}$  a molecule located at  $\mathbf{r}_{\text{mol}}$  will acquire an induced dipole moment

$$\mathbf{p} = \alpha_{\text{pol}} \mathbf{E}_{\text{micro}}(\mathbf{r}_{\text{mol}}) \quad (4.39)$$

where  $\mathbf{E}_{\text{micro}}(\mathbf{r}_{\text{mol}})$  is the *microscopic electric field* at the location of the molecule.

In a slab of dielectric in a uniform external electric field  $\mathbf{E}_{\text{ext}}$ , perpendicular to the slab, the macroscopic field inside the slab will be

$$\mathbf{E}_{\text{macro}} = \frac{\varepsilon_0}{\varepsilon} \mathbf{E}_{\text{ext}}. \quad (4.40)$$

But this is not the electric field we need for Eq. 4.39 — we need the field that would be at  $\mathbf{r}_{\text{mol}}$  if the molecule weren't there.

For amorphous substances we can approximate the microscopic electric field by the field at the centre of a spherical hole of radius  $a$  and volume  $\sim 1/N$  surrounding the molecule, where  $N$  is the number density of molecules (Fig. 4.8). At the centre of the hole, “O”, we need to add the field due to the surface polarisation charge of density  $\sigma_{\text{pol}}(\theta)$  on the surface  $S$  of the hole,

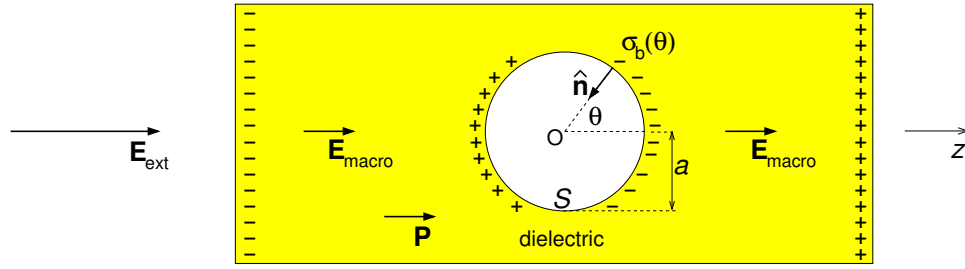
$$\mathbf{E}_{\text{micro}} = \mathbf{E}_{\text{macro}} + \frac{1}{4\pi\varepsilon_0} \oint_S \frac{\sigma_{\text{pol}}(\theta) \hat{\mathbf{R}}}{R^2} dS = \mathbf{E}_{\text{macro}} + \frac{1}{4\pi\varepsilon_0} \oint_S \frac{\sigma_{\text{pol}}(\theta) \hat{\mathbf{n}}}{a^2} dS. \quad (4.41)$$

The surface polarisation charge density is

$$\sigma_{\text{pol}}(\theta) = \mathbf{P} \cdot \hat{\mathbf{n}} = P \hat{\mathbf{z}} \cdot \hat{\mathbf{n}} = -P \cos \theta. \quad (4.42)$$

The minus sign comes from the direction of  $\hat{\mathbf{n}}$  which is outward from the volume of the material (towards centre of hole).





**Figure 4.8:** Geometry used for calculating microscopic electric field inside an amorphous dielectric.

From symmetry arguments  $\mathbf{E}_{\text{micro}}$  must be in the  $z$  direction, and using  $\hat{\mathbf{n}} \cdot \hat{\mathbf{z}} = -\cos \theta$  we have

$$\mathbf{E}_{\text{micro}} \cdot \hat{\mathbf{z}} = E_{\text{macro}} + \frac{1}{4\pi\epsilon_0 a^2} \int_{-1}^1 (-P \cos \theta)(-\cos \theta) 2\pi a^2 d(\cos \theta), \quad (4.43)$$

$$\mathbf{E}_{\text{micro}} \cdot \hat{\mathbf{z}} = E_{\text{macro}} + \frac{P}{2\epsilon_0} \int_{-1}^1 \cos^2 \theta d \cos \theta. \quad (4.44)$$

$$\therefore \mathbf{E}_{\text{micro}} = \mathbf{E}_{\text{macro}} + \frac{\mathbf{P}}{3\epsilon_0}. \quad (4.45)$$

We are now in a position to obtain the Clausius-Mossotti formula, named after German physicist Rudolf Julius Emanuel Clausius (1822–1888) and Italian physicist Ottaviano-Fabrizio Mossotti (1791–1863), which relates the molecular polarisability to the relative permittivity. If the number density of molecules is  $N$ , the polarisation will be

$$\mathbf{P} = N\alpha_{\text{pol}}\mathbf{E}_{\text{micro}} = N\alpha_{\text{pol}}\left(\mathbf{E}_{\text{macro}} + \frac{\mathbf{P}}{3\epsilon_0}\right). \quad (4.46)$$

Solving for  $\mathbf{P}$ ,

$$\mathbf{P} = \frac{N\alpha_{\text{pol}}}{1 - N\alpha_{\text{pol}}/3\epsilon_0} \mathbf{E}_{\text{macro}}. \quad (4.47)$$

Hence, we obtain the susceptibility and the relative permittivity

$$\chi_e = \frac{P}{\epsilon_0 E} = \frac{N\alpha_{\text{pol}}/\epsilon_0}{1 - N\alpha_{\text{pol}}/3\epsilon_0}, \quad (4.48)$$

$$\epsilon_r = (1 + \chi_e) = \frac{1 + 2N\alpha_{\text{pol}}/3\epsilon_0}{1 - N\alpha_{\text{pol}}/3\epsilon_0}, \quad (4.49)$$

and finally the Clausius-Mossotti formula

$$\alpha_{\text{pol}} = \frac{3\varepsilon_0}{N} \left( \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right). \quad (4.50)$$

An example is given in Exercise 4-7

## Summary of Chapter 4

### Macroscopic dielectric theory

- The polarisation field  $\mathbf{P}$  (units:  $\text{C m}^{-2}$ ) is the net dipole moment per unit volume.
- Surface and volume polarisation charge due to polarisation of dielectric

$$\rho_{\text{pol}}(\mathbf{r}) = -\nabla \cdot \mathbf{P}(\mathbf{r}) \quad (\text{unit: } \text{C m}^{-3}), \quad \sigma_{\text{pol}}(\mathbf{r}) = \mathbf{P}(\mathbf{r}) \cdot \hat{\mathbf{n}} \quad (\text{unit: } \text{C m}^{-2}). \quad (4.51)$$

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- Gauss' law and the electric displacement field  $\mathbf{D}(\mathbf{r}) = [\varepsilon_0 \mathbf{E}(\mathbf{r}) + \mathbf{P}(\mathbf{r})]$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho_f(\mathbf{r}), \quad \oint \mathbf{D} \cdot d\mathbf{S} = Q_{f,\text{encl}}. \quad (4.52)$$

- For linear dielectrics  $\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}$

$$\mathbf{D} = (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \varepsilon_0(1 + \chi_e) \mathbf{E} = \varepsilon \mathbf{E} = \varepsilon_r \varepsilon_0 \mathbf{E} = K \varepsilon_0 \mathbf{E} \quad (4.53)$$

where  $\varepsilon$  is the permittivity,  $K = \varepsilon_r$  is the relative permittivity or dielectric constant.

- Boundary conditions at interface between two dielectrics

$$(\mathbf{D}_1 - \mathbf{D}_2) \cdot \hat{\mathbf{n}}_1 = \sigma_f, \quad \mathbf{E}_1^{\parallel} = \mathbf{E}_2^{\parallel}. \quad (4.54)$$

### Microscopic dielectric theory

- In an electric field, atoms or molecules acquire a dipole moment  $\mathbf{p} = \alpha \mathbf{E}_{\text{micro}}$  where  $\alpha_{\text{pol}}$  is the molecular polarisability.
- The “microscopic electric field” at the location of an atom or molecule differs from the macroscopic field in a dielectric. In an amorphous dielectric

$$\mathbf{E}_{\text{micro}} = \mathbf{E}_{\text{macro}} + \frac{\mathbf{P}}{3\varepsilon_0}. \quad (4.55)$$

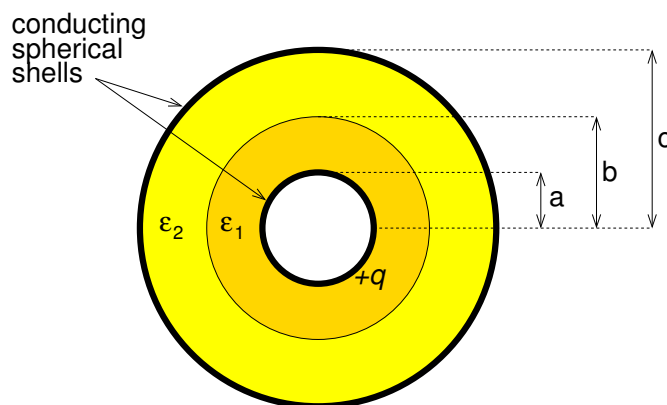
- Clausius-Mossotti formula

$$\alpha_{\text{pol}} = \frac{3\varepsilon_0}{N} \left( \frac{\varepsilon_r - 1}{\varepsilon_r + 2} \right). \quad (4.56)$$

### Exercises on Chapter 4

- 4–1 A dielectric sphere (dielectric constant  $K$ ) of radius  $a$  is placed in an initially uniform electric field  $\mathbf{E}_0$ . (a) What are the boundary conditions on  $V$ ,  $\mathbf{E}$  and  $\mathbf{D}$  for this problem. (b) Find the potential everywhere. (c) Find  $\mathbf{E}$ ,  $\mathbf{D}$  and  $\mathbf{P}$  everywhere. (d) Find the dipole moment of the sphere and the surface polarisation charge density.

- 4–2 An electret, i.e. a piece of material with a permanent electric polarisation, is in the shape of a sphere of radius  $a$  and has  $\mathbf{P}(\mathbf{r}) = \mathbf{P}_0$ . (a) Find the surface polarisation charge density and the dipole moment of the sphere, (b) find  $V$ ,  $\mathbf{E}$ , and  $\mathbf{D}$  everywhere, and (c) sketch the field lines of  $\mathbf{E}$  and  $\mathbf{D}$ .
- 4–3 The space between two concentric conducting cylinders of radius  $a$  and  $b > a$  and length  $L \gg b$  is filled with a dielectric with permittivity  $\varepsilon$ . The inner and outer conductors are held at potentials  $V_a$  and  $V_b$ , respectively. Find: (a)  $\mathbf{E}$ ,  $\mathbf{D}$  and  $\mathbf{P}$  everywhere; (b) the polarisation surface and volume charge density everywhere, and the net polarisation charge; (c) the free charge on the inner and outer conductors, and the capacitance.
- 4–4 A spherical capacitor is filled with two different dielectrics with permittivities  $\varepsilon_1$  and  $\varepsilon_2$  as shown in the diagram. The capacitor is charged such that charge  $+q$  is on the inner conductor. Find: (a)  $\mathbf{D}$ ,  $\mathbf{E}$  and  $\mathbf{P}$  everywhere; (b) the polarisation surface and volume charge density everywhere; (c) the net polarisation charge; (d) the potential difference between the inner and outer conductor, and the capacitance of the capacitor.



- 4–5 A uniform slab of material with relative permittivity  $\varepsilon_r$  is suspended parallel to the  $xy$ -plane, and has its lower surface at  $z = 0$  and its upper surface at  $z = d$ . Outside the slab there is a uniform electric field  $\mathbf{E}_0 = E_0(\sin \theta_0 \hat{\mathbf{x}} - \cos \theta_0 \hat{\mathbf{z}})$ . (a) Find formulae for  $\mathbf{E}$ ,  $\mathbf{D}$  and  $\mathbf{P}$  in the dielectric, the angle between  $\mathbf{E}$  in the dielectric and the normal to the surface, and the surface polarisation charge density at  $z = 0$  and  $z = d$ . (b) Find numerical values for the case of  $E_0 = 1000 \text{ V m}^{-1}$ ,  $\theta_0 = 45^\circ$  and  $\varepsilon_r = 2$ , and include a sketch showing field directions.
- 4–6 Derive the force on an electric dipole in a non-uniform electric field.
- 4–7 The relative permittivities of Nitrogen, Argon and Hydrogen in gas (at  $20^\circ\text{C}$ ) and liquid phases are given below.

Element	N <sub>2</sub> gas	Ar gas	H <sub>2</sub> gas	N <sub>2</sub> liquid	Ar liquid	H <sub>2</sub> liquid
$\epsilon_r$	1.000546	1.000517	1.000272	1.45	1.53	1.22

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Use the Clausius-Mossotti formula to find the electronic polarisability, and compare the results for the same elements in the liquid and gas phases. [You will need to look up any constants and the atomic weights and densities required.]

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## 5 Magnetic field and vector potential

### Learning objectives

- To understand the concept of the magnetic field, and that electric currents cause magnetic fields.
- To learn that an electric charge moving in a magnetic field experiences a force given by the Lorentz force law, and that as a result wires carrying electric currents in a magnetic field experience a force given by Ampère's force law.
- To be able to apply the Biot-Savart law to calculate magnetic fields due to simple current distributions.
- To be able to calculate the vector potential for simple current distributions, and to use it to find the magnetic field.
- To be able to use Ampère's law in integral and differential form to find the current density from magnetic fields, and the magnetic fields from symmetric current distributions.
- To understand the concept of the magnetic flux, that it must be zero for a closed surface (no magnetic charge law), and how the magnetic flux linking two circuits leads to the concept of mutual inductance.
- To learn about the multipole expansion of the vector potential and that the dipole term is the lowest non-zero term.
- To be able to obtain the magnetic dipole moment of a current distribution, and the magnetic field of a magnetic dipole.
- To be able to calculate the torque on a magnetic dipole or a circuit in a magnetic field.

### 5.1 Magnetism

Magnets have been known since antiquity – pieces of lodestone (magnetised magnetite  $\text{Fe}_3\text{O}_4$ ) attract pieces of iron, and when suspended freely point to the magnetic North Pole. Magnetite is a ferromagnetic mineral which became magnetised by strong magnetic fields, possibly associated with lightning strike. Compasses were invented in China in the 2nd Century B.C. and were being used by European navigators by the 12th Century A.D. A compass points in the direction of the *magnetic field*.

Magnetic fields occur throughout the Universe – the Earth’s magnetic field is probably produced by electrical currents associated with magnetohydrodynamic flows within the Earth’s core. Similarly, the Sun and the stars have magnetic fields, pulsars being extreme examples, with some varieties known as magnetars having fields as high as  $10^{14}$  times that on the Earth’s surface. Magnetic fields also pervade the Solar System, interstellar space and, at some level, everywhere in the Universe with the possible exception of giant voids in cosmic structure.

Apart from permanent magnets, the main sources of magnetic fields are electrical currents. In the case of a current along a long straight wire, the magnetic field circulates around the current as shown in Fig. 5.1(a). The unit of magnetic field,  $\mathbf{B}$ , is the tesla (T) which is named after Serbian-American electrical engineer Nicola Tesla (1856–1943).

## 5.2 The magnetic field and forces

As well as producing magnetic fields, electrical currents experience a force in the presence of a magnetic field. For example, the current in one wire produces a magnetic field which causes the current in another wire to experience a force. Historically, this magnetic force between two wires carrying currents was investigated in the 1820’s and 1830’s by Gauss and French physicist André-Marie Ampère (1775–1836). A formulation giving the force on a particle with charge  $Q$  moving with velocity  $\mathbf{v}$  through an electric field  $\mathbf{E}$  and magnetic field  $\mathbf{B}$  is the *Lorentz force* (see Fig. 5.1b),

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \quad (5.1)$$

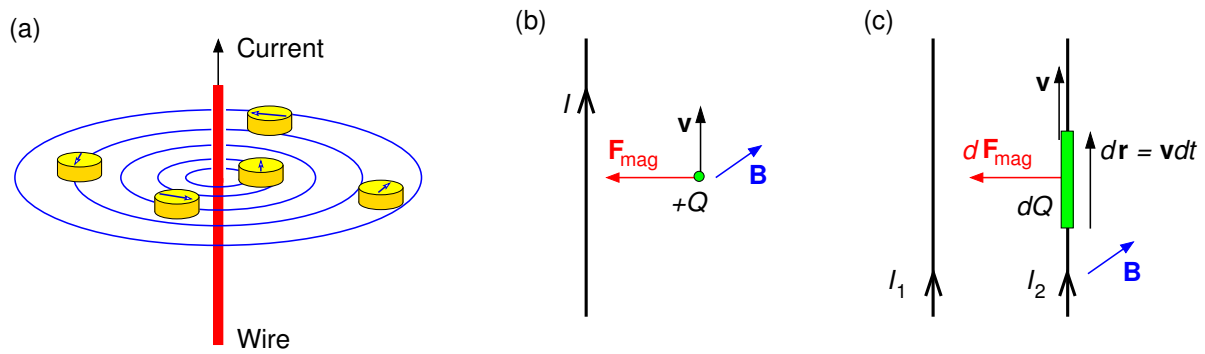
named after Dutch physicist Hendrik Antoon Lorentz (1853–1928), of Lorentz transformation fame.

### 5.2.1 Forces on currents

We can consider the force between two parallel wires using the Lorentz force law as in Fig. 5.1(c). We define a *line current*  $I$  as the net charge per unit time flowing along a filamentary wire, and so the net charge flowing past some point in time  $dt$  is  $dQ = I dt$ . The unit of current is the amp (A) and is named after Ampère. Note that  $1 \text{ A} = 1 \text{ C s}^{-1}$ .

If the effective current flow speed is  $v$  then charge  $dQ$  is located over length  $dr = v dt$  of wire,





**Figure 5.1:** (a) Circulation of magnetic field around a long straight current. (b) Lorentz force on charge  $Q$  moving at velocity  $\mathbf{v}$  parallel to the wire carrying current  $I$ . (c) Lorentz force on length  $dr$  of a second wire (carrying current  $I_2$ ) due to the magnetic field of current  $I_1$ .

and hence the force on section  $dr$  is

$$d\mathbf{F}_{\text{mag}} = dQ\mathbf{v} \times \mathbf{B} = (I dt)\mathbf{v} \times \mathbf{B} = I d\mathbf{r} \times \mathbf{B}. \quad (5.2)$$

Since the force on section  $dr$  of the wire is  $d\mathbf{F}_{\text{mag}} = I d\mathbf{r} \times \mathbf{B}$  the force on the entire circuit is

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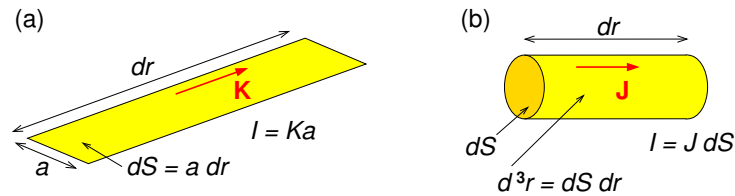


$$\mathbf{F}_{\text{mag}} = \oint (I d\mathbf{r} \times \mathbf{B}). \quad (5.3)$$

This is one version of Ampère's force law.

Currents can also flow across a surface as “current sheets” and are described by the *surface current density*  $\mathbf{K}$  (amps per metre). Currents flowing through a volume are described by the *volume current density*  $\mathbf{J}$  (amps per square metre). The latter two cases are illustrated in Fig. 5.2. Since the force on section  $d\mathbf{r}$  of the wire is  $d\mathbf{F}_{\text{mag}} = I d\mathbf{r} \times \mathbf{B}$  the forces on the entire circuit for the three cases are:

$$\mathbf{F}_{\text{mag}} = \oint I d\mathbf{r} \times \mathbf{B} \longleftrightarrow \int \mathbf{K} \times \mathbf{B} dS \longleftrightarrow \int \mathbf{J} \times \mathbf{B} d^3r. \quad (5.4)$$



**Figure 5.2:** Sketch illustrating (a) surface current density (b) volume current density.

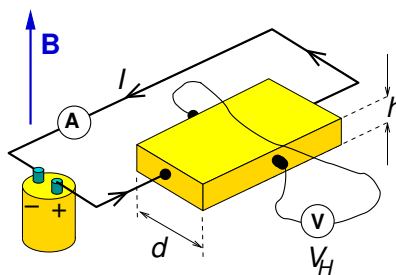
### 5.2.2 Hall effect

As discovered by American physicist Edwin H. Hall (1855-1938), when a current is passed through a semiconductor in the presence of a magnetic field a voltage, called the *Hall voltage*, develops perpendicular to the current direction. This phenomena, due to the Lorentz force on electrons or holes carrying the current, is called the *Hall effect*. The geometry for calculating the Hall voltage is shown in Fig. 5.3.

Note that the direction of the current  $I$  in the diagram is that of the conventional current, so that the motion of electrons is in the opposite direction. The current is

$$I = J h d = (n_c q v) h d, \quad (5.5)$$

where  $J$  is the current density,  $n_c$  the number density of charge carriers of charge  $q$ , and  $v$  their effective velocity. The Lorentz force  $q\mathbf{v} \times \mathbf{B}$  acts on the charge carriers deflecting them towards



**Figure 5.3:** Geometry for calculating the Hall voltage.

one side, and causing one side of the semiconducting strip to become positive and the other negative.

In equilibrium, the Lorentz force of magnitude  $qvB$  is opposed by the electrostatic force of magnitude  $qV_H/d$  due to this charge separation, where  $V_H$  is the Hall voltage. Hence,

$$V_H = d v B = d \frac{I}{n_c q h d} B = \frac{1}{n_c q} \frac{I B}{h} = R_H \frac{I B}{h}. \quad (5.6)$$

The ratio of the Hall voltage to the product of the current and the magnetic field divided by the element thickness is known as the Hall coefficient,

$$R_H = \frac{1}{n_c q}. \quad (5.7)$$

It is a characteristic of the material from which the semiconductor is made, as its value depends on the type, number and properties of the charge carriers: electrons in a conductor ( $q = -e$ ), electrons in a “n-type semiconductor” ( $q = -e$ ), and holes which act like positive electrons in a “p-type semiconductor” ( $q = +e$ ).

The Hall effect can be used to measure magnetic fields with a “Hall probe”. The *quantum Hall effect* which occurs for a two-dimensional electron gas in high magnetic fields and low temperatures was discovered by in 1980 by Klaus von Klitzing (b. 1943) and has led to a new standard for measurement of electrical resistance.

### 5.2.3 Cyclotron motion

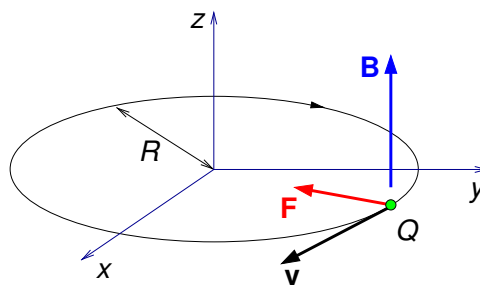
A charged particle with velocity perpendicular to a uniform magnetic field will move in a circular orbit as in Fig. 5.4. The radius of the orbit and its angular frequency can be found by equating

the Lorentz force  $QvB$  to the centripetal force  $mv^2/R$ , giving for non-relativistic velocities

$$R = \frac{mv}{QB}, \quad \omega_{\text{cycl}} = \frac{v}{R} = \frac{QB}{m}, \quad (5.8)$$

where  $\omega_{\text{cycl}}$  is called the *cyclotron frequency*. For relativistic particles of total energy  $\gamma mc^2$  and momentum  $\gamma mv$

$$R = \frac{\gamma mv}{QB}, \quad \omega = \frac{v}{R} = \frac{QB}{\gamma m}. \quad (5.9)$$



**Figure 5.4:** Circular motion of a charged particle in a magnetic field.

If  $\mathbf{v}$  is not perpendicular to  $\mathbf{B}$ , the particle will undergo helical motion.

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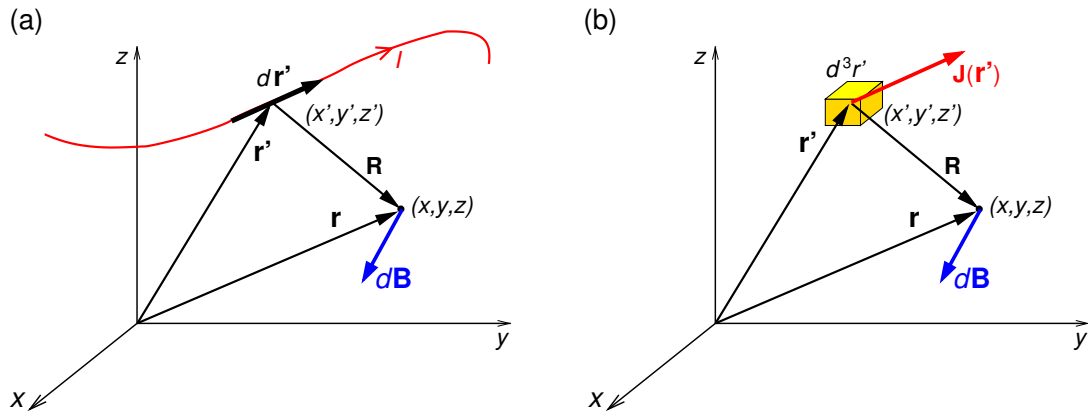
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### 5.2.4 Biot-Savart law

Named after French physicists Jean-Baptiste Biot (1774–1862) and Félix Savart (1791–1841) this experiment-based law gives the magnetic field due to arbitrary line currents (Fig. 5.5a), surface currents and volume current distributions (Fig. 5.5b)

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\mathbf{r}' \times \hat{\mathbf{R}}}{R^2} \longleftrightarrow \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\mathbf{R}}}{R^2} dS' \longleftrightarrow \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathbf{R}}}{R^2} d^3r', \quad (5.10)$$

where  $\mathbf{R} = (\mathbf{r} - \mathbf{r}')$  and  $\mu_0 = 4\pi \times 10^{-7} \text{ H m}^{-1}$  is the *permeability of free space* — the henry (H) is the unit of inductance and is named after American scientist Joseph Henry (1797–1878).



**Figure 5.5:** Contribution  $d\mathbf{B}$  of (a) section  $d\mathbf{r}'$  of line current  $I$ , and (b) volume current density  $\mathbf{J}(\mathbf{r}')$  inside volume  $d^3r'$ , to magnetic field at  $\mathbf{r}$ .

We can use the Biot-Savart law to find the magnetic field of a long straight wire carrying current  $I$  as in Fig. 5.6. From the cross product,  $d\mathbf{B}$  must be into the page as shown, i.e. in the  $\hat{\phi}$  direction in cylindrical coordinates  $(\rho, \phi, z)$ . The contribution to the magnetic field at  $(\rho, \phi, 0)$

due to the section of wire of length  $dz'$  at  $z'$  is

$$d\mathbf{B}(\rho, \phi, 0) = \frac{\mu_0 I}{4\pi} \frac{dz' \times \hat{\mathbf{R}}}{R^2}. \quad (5.11)$$

$$\therefore \mathbf{B}(\rho, \phi, 0) = \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{\sin \theta}{R^2} dz' \hat{\phi}, \quad (5.12)$$

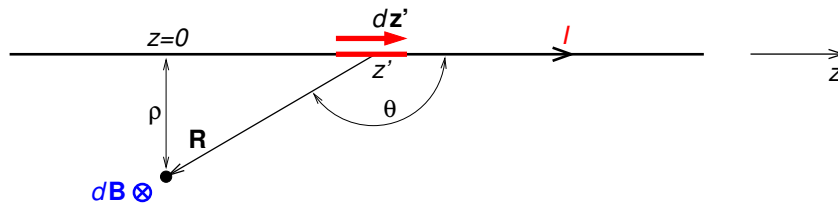
$$= \frac{\mu_0 I}{4\pi} \int_{-\infty}^{\infty} \frac{\rho}{R} \frac{1}{R^2} dz' \hat{\phi}, \quad (5.13)$$

$$= \frac{\mu_0 I}{4\pi} \rho \int_{-\infty}^{\infty} \frac{1}{(\rho^2 + z'^2)^{3/2}} dz' \hat{\phi}, \quad (5.14)$$

$$\mathbf{B}(\rho, \phi, 0) = \frac{\mu_0 I}{4\pi} \rho \left[ \frac{z'}{\rho^2 \sqrt{\rho^2 + z'^2}} \right]_{-\infty}^{\infty} \hat{\phi}. \quad (5.15)$$

$$\therefore \mathbf{B}(\rho, \phi, z) = \frac{\mu_0 I}{2\pi \rho} \hat{\phi} \quad (5.16)$$

since the magnitude can only depend on  $\rho$  and  $I$ , and  $\mathbf{B}$  is in the  $\hat{\phi}$  direction as in Fig. 5.1(a).



**Figure 5.6:** Geometry for calculation of the magnetic field due to a long straight wire carrying current  $I$ .

### 5.3 The magnetic vector potential

From Biot and Savart's law we can find a potential which is a vector field, called the *magnetic vector potential*  $\mathbf{A}(\mathbf{r})$ , from which we can obtain the magnetic field  $\mathbf{B}(\mathbf{r})$  in an analogous (but different) way to obtaining the electric field from the electrostatic potential. We start with the Biot-Savart law, and use an identity for  $\nabla R^{-1}$  (Eq. B.7),

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \times \frac{\hat{\mathbf{R}}}{R^2} d^3r' = -\frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \times \nabla \left( \frac{1}{R} \right) d^3r'. \quad (5.17)$$

Then, using the product rule  $\nabla \times (u\mathbf{F}) = (\nabla u) \times \mathbf{F} + u(\nabla \times \mathbf{F})$  we have

$$-\mathbf{J}(\mathbf{r}') \times \nabla \left( \frac{1}{R} \right) = \nabla \times \left( \frac{\mathbf{J}(\mathbf{r}')}{R} \right) - \frac{1}{R} [\nabla \times \mathbf{J}(\mathbf{r}')]. \quad (5.18)$$

But  $\nabla \times \mathbf{J}(\mathbf{r}') = 0$  because here the differentiation is with respect to  $x, y, z$  (not  $x', y', z'$ ), and so

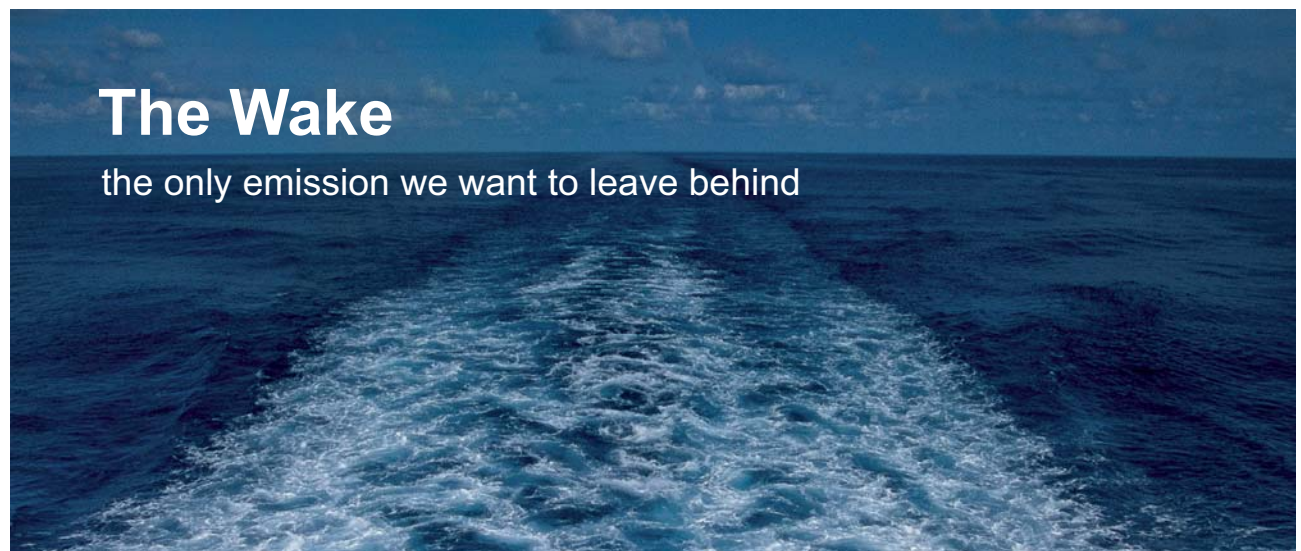
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \nabla \times \left( \frac{\mathbf{J}(\mathbf{r}')}{R} \right) d^3r' = \nabla \times \left( \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{R} d^3r' \right). \quad (5.19)$$

$$\therefore \mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}), \quad (5.20)$$

where

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{R} d^3r' \quad (5.21)$$

is the magnetic vector potential, or just the “vector potential”.



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Exercise 5–2 is to use Eq. 5.21 to find the vector potential of an infinite straight wire along the  $z$  axis carrying current  $I$ , although we can actually find this in a simpler way from the magnetic field which we know to be  $\mu_0 I / 2\pi\rho \hat{\phi}$  (Eq. 5.16), and we will do so now.

Using  $\mathbf{B} = \nabla \times \mathbf{A}$  in cylindrical coordinates

$$\frac{\mu_0 I}{2\pi\rho} \hat{\phi} = \left( \frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{\rho} + \left( \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \hat{\phi} + \frac{1}{\rho} \left( \frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right) \hat{z}. \quad (5.22)$$

$$\therefore \frac{\mu_0 I}{2\pi\rho} = -\frac{\partial A_z}{\partial \rho} \quad (5.23)$$

$$\therefore \int dA_z = -\frac{\mu_0 I}{2\pi} \int \frac{d\rho}{\rho} \quad (5.24)$$

$$\therefore \mathbf{A}(\mathbf{r}) = -\hat{z} \frac{\mu_0 I}{2\pi} \ln(\rho) + \hat{z} C \quad (5.25)$$

where  $C$  is an integration constant whose value has no effect on the magnetic field.

### 5.3.1 Gauge transformations

The magnetic field is unchanged by adding the gradient of an arbitrary scalar field  $U(\mathbf{r})$  to  $\mathbf{A}$ ,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{R} d^3 r' + \nabla U(\mathbf{r}), \quad (5.26)$$

because  $\nabla \times \nabla U(\mathbf{r}) = 0$ . The transformation  $\mathbf{A} \rightarrow \mathbf{A} + \nabla U$  is called a *gauge transformation*. Exercise 5–3 is to show that

$$\nabla \cdot \mathbf{A}(\mathbf{r}) = \nabla^2 U(\mathbf{r}). \quad (5.27)$$

The “gauge” is determined by the value of  $\nabla \cdot \mathbf{A}$ , and in magnetostatics, it is convenient to choose  $\nabla \cdot \mathbf{A} = 0$  (Coulomb gauge).

## 5.4 Ampère's law

This law, named after André-Marie Ampère, is obtained by taking the curl of  $\mathbf{B}$  and using the 2nd derivative rule for curl of curl, followed by  $\nabla \cdot \mathbf{A} = 0$

$$\nabla \times \mathbf{B}(\mathbf{r}) = \nabla \times [\nabla \times \mathbf{A}(\mathbf{r})] \quad (5.28)$$

$$= \nabla [\nabla \cdot \mathbf{A}(\mathbf{r})] - \nabla^2 \mathbf{A}(\mathbf{r}) \quad (5.29)$$

$$= 0 - \nabla^2 \left[ \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{R} d^3 r' \right] \quad (5.30)$$

$$= -\frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') \nabla^2 \left( \frac{1}{R} \right) d^3 r'. \quad (5.31)$$

Note that in the last line the Laplacian operator has moved to the right of  $\mathbf{J}(\mathbf{r}')$  because the differentiation here is with respect to unprimed coordinates. Now, Eq. C.39 gives

$$\nabla^2 \left( \frac{1}{R} \right) = -4\pi \delta^3(\mathbf{R}) = -4\pi \delta^3(\mathbf{r} - \mathbf{r}'). \quad (5.32)$$

$$\therefore \nabla \times \mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') [-4\pi \delta^3(\mathbf{r} - \mathbf{r}')] d^3 r'. \quad (5.33)$$

$$\therefore \nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}), \quad (5.34)$$

which is Ampère's law in differential form. We can use Stokes' theorem (Appendix C.6) to find Ampère's law also in integral form

$$\oint_{\Gamma} \mathbf{B}(\mathbf{r}) \cdot d\mathbf{r} = \mu_0 \int_S \mathbf{J}(\mathbf{r}) \cdot d\mathbf{S} \quad (5.35)$$

where loop  $\Gamma$  bounds surface  $S$ .

Ampère's law in integral form can be exploited in a similar way to Gauss' law in integral form if the symmetry of problem allows. An “Ampèrian loop” plays an equivalent role to the “Gaussian surface” when using Gauss law. As an example, consider the magnetic field of a long straight wire carrying current  $I$ . There is cylindrical symmetry, and we can for convenience define the  $z$  axis to be along the wire. Then the magnitude of the field must depend only on cylindrical coordinate  $\rho$  and the field must be in the  $\hat{\phi}$  direction as in Fig. 5.1(a). Taking a circular

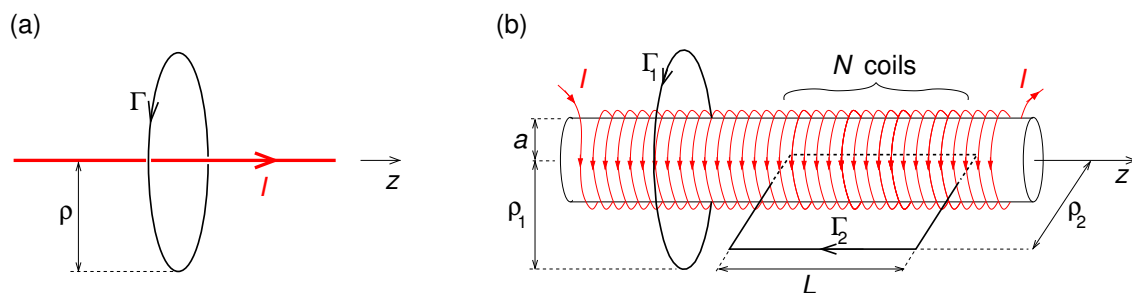


Amperian loop  $\Gamma$  of radius  $\rho$  as in Fig. 5.7(a)

$$\oint_{\Gamma} \mathbf{B} \cdot d\mathbf{r} = \mu_0 I_{\text{encl}}, \quad B 2\pi\rho = \mu_0 I, \quad (5.36)$$

giving

$$\mathbf{B} = \frac{\mu_0 I}{2\pi\rho} \hat{\phi}. \quad (5.37)$$



**Figure 5.7:** (a) Amperian loop  $\Gamma$  centred on the axis of a long wire carrying current  $I$ . (b) A long solenoid (much longer than shown) and a circular Amperian loop  $\Gamma_1$  concentric the solenoid's axis, and a rectangular Amperian loop  $\Gamma_2$  with two sides parallel to the solenoid's axis, one of which is inside the solenoid.

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Another example is the long solenoid as depicted in Fig. 5.7(b). For a circular Ampèrian loop  $\Gamma_1$  concentric with the axis,  $I_{\text{encl}} = I$  if  $\Gamma_1$  is outside the solenoid ( $\rho_1 > a$ ), and  $I_{\text{encl}} = 0$  if it is inside ( $\rho_1 < a$ ). Hence, outside the solenoid, the azimuthal component is  $B_\phi(\rho > a, \phi, z) = \mu_0 I / 2\pi\rho$ , whereas inside  $B_\phi(\rho < a, \phi, z) = 0$ . It is widely stated that the magnetic field outside a long solenoid is zero, which appears to contradict this result. It is true that the magnetic field outside is very much less than inside, and one can design a solenoid with  $\mathbf{B} = 0$  outside by having a double coil with current flowing out in the  $+z$  direction in one coil, and back in the  $-z$  direction in the other, such that the net current through the circular loop  $\Gamma_1$  is zero.

From the symmetry of the problem, if there were a component of  $\mathbf{B}$  in the  $\hat{\rho}$  direction it would be the same everywhere over the surface of a concentric cylinder, and this would give  $\oint \mathbf{B} \cdot d\mathbf{S} \neq 0$  which is not allowed if, as it appears, magnetic charge doesn't exist (as is discussed in the next section). We have already found  $B_\phi$  inside and outside the cylinder, so now we examine  $B_z$  using rectangular Amperian loop  $\Gamma_2$  with two sides parallel to the solenoid's axis, one of which is inside the solenoid. The current enclosed is  $I_{\text{encl}} = NI$ , for all  $\rho_2$  values, so that Ampère's law implies that  $B_z L$ , and thus  $B_z$ , is the same for all  $\rho > a$ . But  $B_z(\rho = \infty, \phi, z)$  must be zero, so that  $B_z(\rho > a) = 0$ . Similar arguments show that  $B_z(\rho < a, \phi, z) = B_0$ , i.e.  $B_z$  is constant inside the solenoid. Ampère's law for loop  $\Gamma_2$  is then

$$\oint \mathbf{B} \cdot d\mathbf{r} = \mu_0 I_{\text{encl}}, \quad B_z L = \mu_0 NI, \quad (5.38)$$

giving

$$\mathbf{B}(\rho, \phi, z) = \begin{cases} \mu_0 n I \hat{\mathbf{z}} & \text{inside solenoid} \\ \mu_0 I / (2\pi\rho) \hat{\phi} & \text{outside solenoid (usually neglected)} \end{cases} \quad (5.39)$$

where  $n = N/L$  is the number of coils (or “turns”) per unit length.

## 5.5 Magnetic flux

The magnetic flux through surface  $S$  is defined as

$$\Phi_B \equiv \int_S \mathbf{B}(\mathbf{r}) \cdot d\mathbf{S} = \int_S \nabla \times \mathbf{A}(\mathbf{r}) \cdot d\mathbf{S}, \quad (5.40)$$

and using Stokes' theorem we find that

$$\Phi_B = \oint_{\Gamma} \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r} \quad (5.41)$$

where loop  $\Gamma$  bounds surface  $S$ . This result is very useful for finding the vector potential from the magnetic field in cases where we can exploit the symmetry of the problem, in a similar way to finding the magnetic field from the current density using Ampère's law.

We shall now use Eq. 5.41 to find the vector potential due to current  $I$  flowing in a long solenoid (Fig. 5.7b). From Eq. 5.21, one expects there to be a  $z$  component of  $\mathbf{A}$  outside the solenoid due to the net current  $I$  flowing along the solenoid, but we will neglect this here. Since the current in the solenoid circulates in the  $\hat{\phi}$  direction, from symmetry arguments we should expect that  $\mathbf{A}$  will have a  $\phi$  component. Applying Eq. 5.41 to loop  $\Gamma_1$  for a solenoid of radius  $a$  as shown,

$$2\pi\rho_1 A_{\phi}(\rho_1, \phi, z) = \begin{cases} \pi\rho_1^2\mu_0 nI & \text{inside solenoid } (\rho_1 < a), \\ \pi a^2\mu_0 nI & \text{outside solenoid } (\rho_1 > a). \end{cases} \quad (5.42)$$

Hence,

$$\mathbf{A}(\rho, \phi, z) = \begin{cases} (\mu_0 n I \rho / 2) \hat{\phi} & \text{inside solenoid } (\rho < a), \\ (\mu_0 n I a^2 / 2\rho) \hat{\phi} & \text{outside solenoid } (\rho > a). \end{cases} \quad (5.43)$$

### 5.5.1 No magnetic charge law

We can derive this very important law from

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}), \quad (5.44)$$

and then using the second-derivative rule  $\nabla \cdot (\nabla \times \mathbf{c}) = 0$  we obtain,

$$\nabla \cdot \mathbf{B} = 0. \quad (5.45)$$

Comparing this with Gauss' law in differential form  $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$  we see that Eq 5.45 implies that magnetic charge does not exist. Some particle theories predict the existence of massive magnetic monopoles, but so far none have been found. Magnetic dipoles do exist however, and

are very important in discussing magnetic materials.

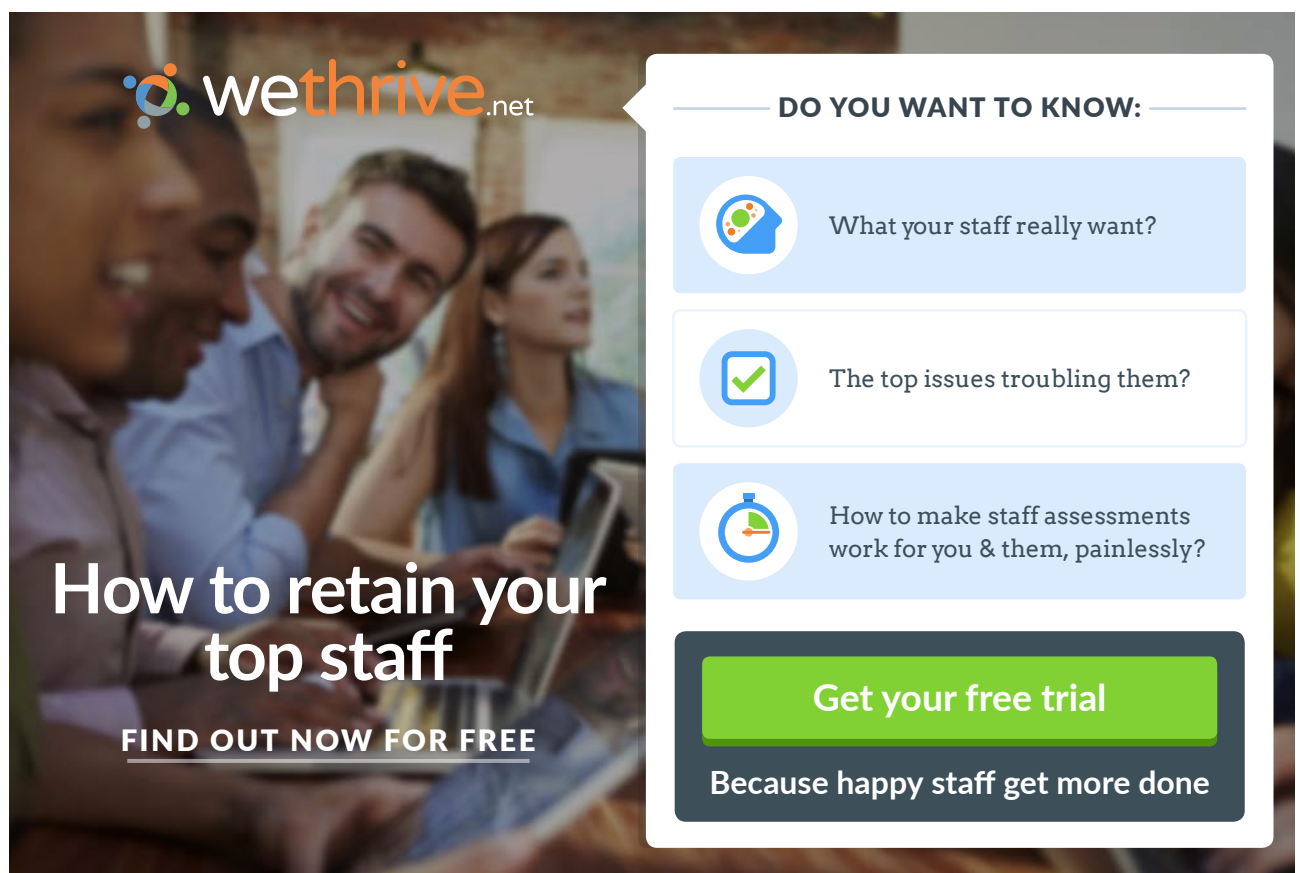
As we shall see later in this chapter, Eq 5.45 given in integral form below (as can be found by applying Gauss' theorem to the differential form),

$$\oint \mathbf{B} \cdot d\mathbf{S} = 0, \quad (5.46)$$

is important in determining the boundary condition on  $\mathbf{B}$  between two materials.

### 5.5.2 Mutual inductance

A current flowing around a loop of wire (e.g. Coil 1 in Fig. 5.8) produces a magnetic field, and some of the field lines may pass through a second loop of wire (e.g. Coil 2). If we double the current  $I_1$  the magnetic field will double, as will the number of field lines through Coil 2. The way to quantify this threading of Coil 1's field through Coil 2 is by the *mutual inductance*  $M$  defined as the magnetic flux through Coil 2 due to current  $I_1$ , divided by  $I_1$



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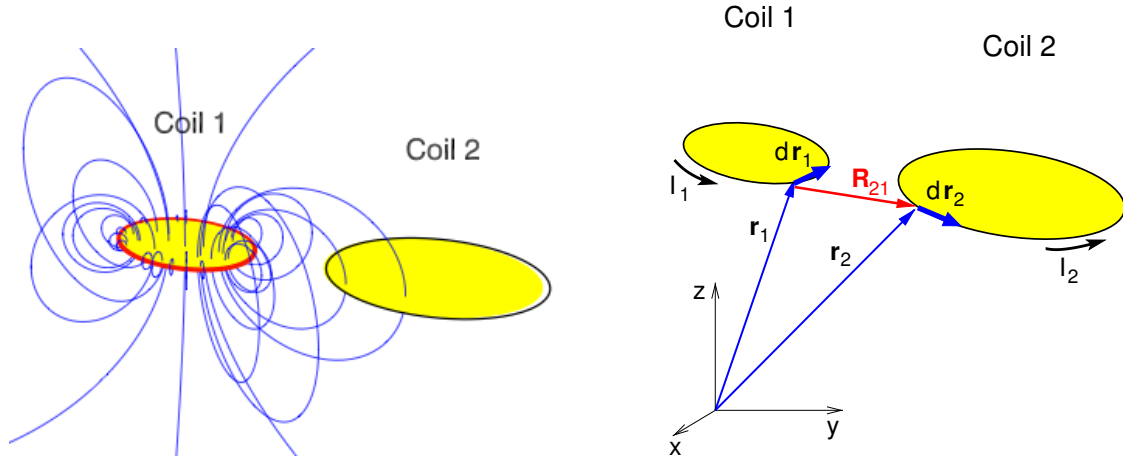
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$$M \equiv \frac{\Phi_{B,2}}{I_1} = \frac{1}{I_1} \oint_2 \mathbf{A}_1(\mathbf{r}_2) \cdot d\mathbf{r}_2 = \frac{1}{I_1} \oint_2 \frac{\mu_0 I_1}{4\pi} \oint_1 \frac{d\mathbf{r}_1}{R_{21}} \cdot d\mathbf{r}_2 \quad (5.47)$$

where  $\mathbf{R}_{21}$  is the separation vector from line element  $d\mathbf{r}_1$  to line element  $d\mathbf{r}_2$  (Fig. 5.8b). Hence

$$M = \frac{\mu_0}{4\pi} \oint_2 \oint_1 \frac{d\mathbf{r}_1 \cdot d\mathbf{r}_2}{R_{21}}, \quad (5.48)$$

and we see that the mutual inductance could be calculated also from magnetic flux due to Coil 2 through Coil 1 divided by  $I_2$  with exactly the same value, i.e.  $M = \Phi_{B,2}/I_1 = \Phi_{B,1}/I_2$ . Also note that  $M$  only depends on the geometry (separation, shapes and sizes) of the two coils. The unit of inductance is the henry (H). We shall return to the topic of inductance in “Essential Electrodynamics” in the context of Faraday’s law.



**Figure 5.8:** (a)–Left: randomly selected magnetic field lines due to current  $I_1$  in Coil 1, some of which thread Coil 2. (b)–Right: geometry for calculation of the mutual inductance of the two coils.

## 5.6 Multipole expansion of vector potential

For the localised current distribution  $\mathbf{J}(\mathbf{r})$  the vector potential and its components are

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{R} d^3r', \quad A_i(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{J_i(\mathbf{r}')}{R} d^3r', \quad (5.49)$$

where  $i = 1, 2, 3$  correspond to  $x, y, z$ . This is similar to Coulomb's law for a localised charge distribution which we expressed as a multipole expansion (Eq. 3.5). Thus

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi r} \int J_i(\mathbf{r}') d^3r' + \frac{\mu_0}{4\pi r^3} \int J_i(\mathbf{r}') \mathbf{r} \cdot \mathbf{r}' d^3r' + \dots \quad (5.50)$$

$$\therefore \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi r} \int \mathbf{J}(\mathbf{r}') d^3r' + \frac{\mu_0}{4\pi r^3} \int \mathbf{J}(\mathbf{r}') \mathbf{r} \cdot \mathbf{r}' d^3r' + \dots \quad (5.51)$$

$$\text{(monopole term)} \qquad \qquad \text{(dipole term)} \quad (5.52)$$

In magnetostatics a localised current distribution may be considered as being made of  $n$  *closed* circuits,  $\Gamma_i$ , carrying currents,  $I_i$ , and so

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi r} \sum_{i=1}^n I_i \oint_{\Gamma_i} d\mathbf{r}' + \frac{\mu_0}{4\pi r^3} \sum_{i=1}^n I_i \oint_{\Gamma_i} d\mathbf{r}' \mathbf{r} \cdot \mathbf{r}' + \dots \quad (5.53)$$

$$\text{(monopole term)} \qquad \qquad \text{(dipole term)}. \quad (5.54)$$

Note, the monopole term vanishes because  $d\mathbf{r}'$  is an exact differential, and we are left with the dipole and higher terms. In Appendix H we show that the dipole term can be written

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}, \quad (5.55)$$

where the *magnetic dipole moment* is

$$\mathbf{m} = \frac{1}{2} \sum_{i=1}^n I_i \oint_{\Gamma_i} \mathbf{r}' \times d\mathbf{r}' \quad (n \text{ current loops}), \quad (5.56)$$

$$\mathbf{m} = \frac{1}{2} \int \mathbf{r}' \times \mathbf{J}(\mathbf{r}') d^3r' \quad (\text{volume current distribution}). \quad (5.57)$$

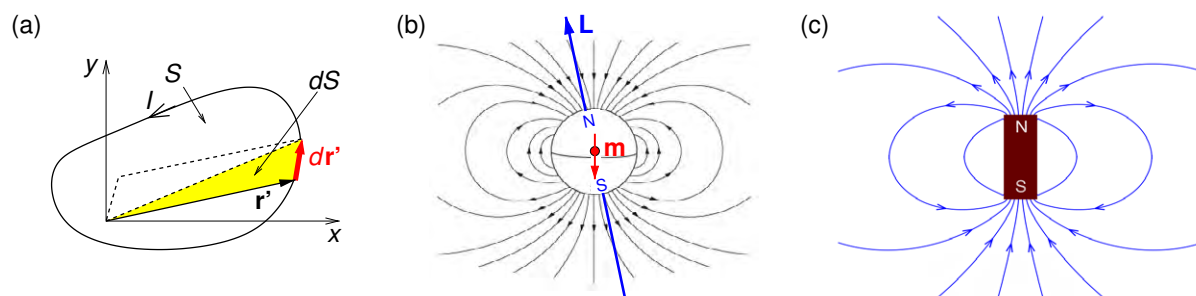
Consider the current loop in the  $x$ - $y$  plane shown in Fig. 5.9(a). The cross product of the two vectors  $\mathbf{r}'$  and  $d\mathbf{r}'$  generates a parallelogram of area  $|\mathbf{r}' \times d\mathbf{r}'|$ , with half its area corresponding to the area of the shaded triangle in Fig. 5.9(a) labelled  $dS$ , i.e.

$$\frac{1}{2} \mathbf{r}' \times d\mathbf{r}' = dS \hat{\mathbf{z}}. \quad (5.58)$$

Hence, integrating Eq. 5.56 around this loop would give

$$\mathbf{m} = IS\hat{\mathbf{z}} \quad (5.59)$$

where  $S$  is the area of the loop.



**Figure 5.9:** (a) Current loop in  $xy$  plane. (b) Earth's magnetic field outside the Earth approximated by that of magnetic dipole  $\mathbf{m}$  at its centre, currently misaligned by  $168.5^\circ$  with respect to the Earth's spin angular momentum  $\mathbf{L}$ . (c) Magnetic field of a bar magnet.

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### 5.6.1 Dipole magnetic field

The magnetic field of a magnetic dipole resembles the electric field of an electric dipole (Fig. 3.3). The Earth's magnetic field outside the Earth's surface can be approximated by that of a dipole of moment  $|\mathbf{m}| = 8.2 \times 10^{22} \text{ A m}^2$  at the centre of the Earth (Fig. 5.9b). The dipole moment  $\mathbf{m}$  is directed at an angle of about  $168.5^\circ$  to Earth's spin direction, so that the “real” magnetic North Pole is actually  $11.5^\circ$  from the geographic *South* Pole! The magnetic field of a bar magnet is sketched in Fig. 5.9(c) showing the magnet's north and south poles – by convention magnetic field lines leave the magnet at its north pole and enter the magnet at its south pole.

We derive the magnetic field of a dipole here using vector identities. A derivation using index notation is carried out in Exercise 5–7. Taking the curl of the vector potential in Eq. 5.55,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \nabla \times \left( \mathbf{m} \times \frac{\mathbf{r}}{r^3} \right). \quad (5.60)$$

Using the product rule for the curl of a cross product,

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\nabla \cdot \mathbf{b})\mathbf{a} - (\nabla \cdot \mathbf{a})\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}, \quad (5.61)$$

we have

$$\nabla \times \left( \mathbf{m} \times \frac{\mathbf{r}}{r^3} \right) = \left( \nabla \cdot \frac{\mathbf{r}}{r^3} \right) \mathbf{m} - (\nabla \cdot \mathbf{m}) \frac{\mathbf{r}}{r^3} + \left( \frac{\mathbf{r}}{r^3} \cdot \nabla \right) \mathbf{m} - (\mathbf{m} \cdot \nabla) \frac{\mathbf{r}}{r^3}, \quad (5.62)$$

$$= 0 - 0 + 0 - (\mathbf{m} \cdot \nabla) \frac{\mathbf{r}}{r^3}, \quad (5.63)$$

where we have used  $\nabla \cdot (\hat{\mathbf{r}}/r^3) = 0$  ( $r > 0$ ) for the 1st term, and that  $\mathbf{m}$  is a constant vector in the 2nd and 3rd terms on the right. Then

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{4\pi} (\mathbf{m} \cdot \nabla) \frac{\mathbf{r}}{r^3}, \quad (5.64)$$

$$= -\frac{\mu_0}{4\pi} \left[ \mathbf{r} (\mathbf{m} \cdot \nabla) \frac{1}{r^3} + \frac{1}{r^3} (\mathbf{m} \cdot \nabla) \mathbf{r} \right]. \quad (5.65)$$



Now, firstly

$$\mathbf{r}(\mathbf{m} \cdot \nabla) \frac{1}{r^3} = \mathbf{r} \left( m_x \frac{\partial}{\partial x} + m_y \frac{\partial}{\partial y} + m_z \frac{\partial}{\partial z} \right) (x^2 + y^2 + z^2)^{-3/2}, \quad (5.66)$$

$$\begin{aligned} &= \mathbf{r} \left[ m_x(-3/2)(x^2 + y^2 + z^2)^{-5/2} 2x + m_y(-3/2)(x^2 + y^2 + z^2)^{-5/2} 2y \right. \\ &\quad \left. + m_z(-3/2)(x^2 + y^2 + z^2)^{-5/2} 2z \right], \end{aligned} \quad (5.67)$$

$$\therefore \mathbf{r}(\mathbf{m} \cdot \nabla) \frac{1}{r^3} = -3(\mathbf{m} \cdot \mathbf{r}) \mathbf{r} \frac{1}{r^5}. \quad (5.68)$$

Secondly,

$$\frac{1}{r^3}(\mathbf{m} \cdot \nabla) \mathbf{r} = \frac{1}{r^3} \left( m_x \frac{\partial}{\partial x} + m_y \frac{\partial}{\partial y} + m_z \frac{\partial}{\partial z} \right) (x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}), \quad (5.69)$$

$$\therefore \frac{1}{r^3}(\mathbf{m} \cdot \nabla) \mathbf{r} = \frac{1}{r^3} \mathbf{m}. \quad (5.70)$$

Hence, for  $r > 0$  the magnetic field of a magnetic dipole of moment  $\mathbf{m}$  is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left[ \frac{3\mathbf{r}(\mathbf{m} \cdot \mathbf{r}) - r^2 \mathbf{m}}{r^5} \right]. \quad (5.71)$$

This has identical form to the electric field due to electric dipole of moment  $\mathbf{p}$  (Eq. 3.19).

### 5.6.2 Torques on magnetic dipoles

The net force on a current loop (or dipole) in a magnetic field is

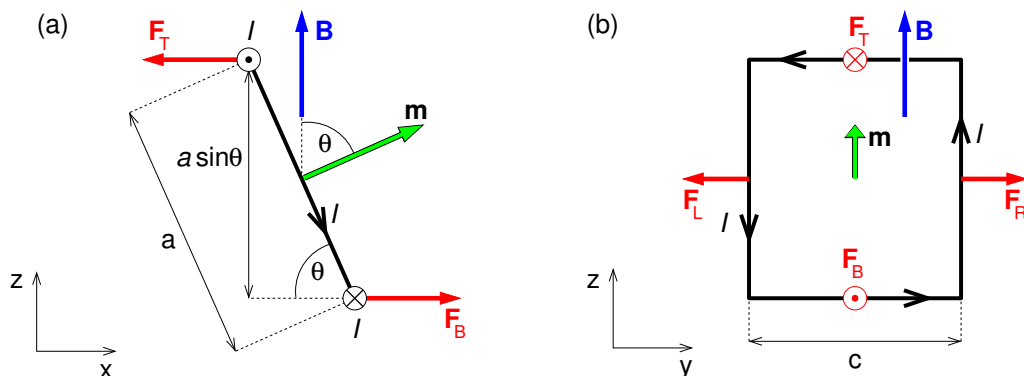
$$\mathbf{F}_{\text{mag}} = I \oint (d\mathbf{r} \times \mathbf{B}), \quad (5.72)$$

which for a uniform magnetic field is

$$\mathbf{F}_{\text{mag}} = I \left( \oint d\mathbf{r} \right) \times \mathbf{B} = 0, \quad \text{because } \oint d\mathbf{r} = 0. \quad (5.73)$$

Although there is no net force, a dipole does experience a torque tending to align its dipole moment with the magnetic field direction. To find the value of the torque consider a rectangular current loop, with sides of length  $a$  and  $c$ , carrying current  $I$  in a uniform magnetic field as shown

in the two orthogonal views in Fig. 5.10(a) and (b). Using Ampère's force law  $d\mathbf{F} = I d\mathbf{r} \times \mathbf{B}$  we will find the force on each of the four sections.



**Figure 5.10:** Geometry for calculating the torque on a rectangular current loop in the presence of a uniform magnetic field (a) looking in the  $\hat{y}$  direction, and (b) looking in the  $-\hat{x}$  direction.

From Fig. 5.10(a), we see the top and bottom sections of the loop are perpendicular to  $\mathbf{B}$ , and so  $\mathbf{F}_T = -IcB\hat{x}$  and  $\mathbf{F}_B = IcB\hat{x} = -\mathbf{F}_T$  and these two forces cancel each other. Similarly, looking at Fig. 5.10(b), we see that the forces on the left and right sections are  $\mathbf{F}_L = -Ia\sin(\pi - \theta)B\hat{y}$  and  $\mathbf{F}_R = Ia\sin(\pi - \theta)B\hat{y} = -\mathbf{F}_L$ , and so also cancel each other such that the net force is zero in agreement with Eq. 5.73.

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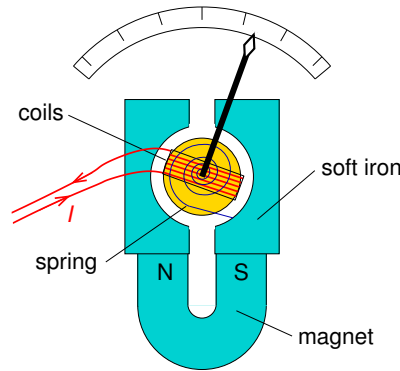
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From Fig. 5.10(a), we see that there will be a torque on the current loop due to forces  $\mathbf{F}_T$  and  $\mathbf{F}_B$  equal to  $\mathbf{N} = -|\mathbf{F}_T|a \sin \theta \hat{\mathbf{y}} = -IacB \sin \theta \hat{\mathbf{y}}$ , and this tends to align  $\mathbf{m}$  parallel to  $\mathbf{B}$ . Given that  $\mathbf{B} = B \hat{\mathbf{z}}$  and  $\mathbf{m} = Iac(\cos \theta \hat{\mathbf{z}} + \sin \theta \hat{\mathbf{x}})$  we find that

$$\mathbf{N} = \mathbf{m} \times \mathbf{B}. \quad (5.74)$$

Such current loops in a magnetic field form the basic design of the *galvanometer* at the heart of all analogue current, voltage or charge measuring devices (Fig. 5.11), and also electric motors. Similarly, these torques also tend to align electron magnetic moments (associated with spin) parallel to  $\mathbf{B}$  causing *paramagnetism*, to be discussed in the next chapter.



**Figure 5.11:** A simple galvanometer to measure current  $I$  has coils through which the current flows, a permanent magnet with shaped pieces to provide the same field configuration with respect to the coils at all deflection angles, a spring to provide a torque to balance the torque on the magnetic moment of the current through the coils, and a needle pointer.

To rotate the dipole moment, work must be done against the torque, and so the potential energy of a dipole in a magnetic field is

$$W(\theta) = \int_{\theta_{\text{ref}}}^{\theta} N(\theta') d\theta' = mB \int_{\theta_{\text{ref}}}^{\theta} \sin \theta' d\theta' = mB(\cos \theta_{\text{ref}} - \cos \theta), \quad (5.75)$$

$$\therefore W = -\mathbf{m} \cdot \mathbf{B} \quad (5.76)$$

where, for convenience, we have chosen the reference angle to be  $\theta_{\text{ref}} = \pi/2$ .

As an aside, although for a uniform magnetic field there is no net force on a magnetic dipole, in a non-uniform magnetic field there is. The force on the dipole is

$$\mathbf{F} = -\nabla W = \nabla(\mathbf{m} \cdot \mathbf{B}). \quad (5.77)$$

If the dipole is aligned with  $\mathbf{B}$ , it will experience a force in the direction of increasing  $\mathbf{B}$ . Actually, the result in Eq. 5.77 glosses over a number of issues including whether the dipole is due to a current loop, electron orbital motion or spin. Nevertheless, Eq. 5.77 explains why paramagnets and ferromagnets are attracted to strong magnetic fields whereas diamagnets are repelled.

## Summary of Chapter 5

### Magnetic forces

- Lorentz force law  $\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ .
- Force on a circuit in a magnetic field  $\mathbf{F}_{\text{mag}} = \oint (I d\mathbf{r} \times \mathbf{B})$ .
- For surface and volume currents  $\mathbf{F}_{\text{mag}} = \int \mathbf{K} \times \mathbf{B} dS = \int \mathbf{J} \times \mathbf{B} d^3r$ .

### Magnetic field and vector potential

- Biot-Savart law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \oint \frac{d\mathbf{r}' \times \hat{\mathbf{R}}}{R^2}, \quad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathbf{R}}}{R^2} d^3r'. \quad (5.78)$$

- Vector potential  $\mathbf{A}$  where  $\mathbf{B} = \nabla \times \mathbf{A}$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{R} d^3r'. \quad (5.79)$$

### Ampère's law and the no magnetic charge law

- Ampère's law

$$\nabla \times \mathbf{B}(\mathbf{r}) = \mu_0 \mathbf{J}(\mathbf{r}), \quad \oint_{\Gamma} \mathbf{B}(\mathbf{r}) \cdot d\mathbf{r} = \mu_0 \int_S \mathbf{J}(\mathbf{r}) \cdot d\mathbf{S}, \quad \oint \mathbf{B} \cdot d\mathbf{r} = \mu_0 I_{\text{encl}}. \quad (5.80)$$

- Magnetic flux  $\Phi_B \equiv \int_S \mathbf{B}(\mathbf{r}) \cdot d\mathbf{S} = \oint_{\Gamma} \mathbf{A}(\mathbf{r}) \cdot d\mathbf{r}$ .

- No magnetic charge law  $\nabla \cdot \mathbf{B} = 0, \quad \oint \mathbf{B} \cdot d\mathbf{S} = 0$ .

**Multipole expansion of vector potential**

- The monopole term is zero, and the dipole term is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}. \quad (5.81)$$

- Magnetic dipole moment for  $n$  current loops and for a volume current distribution

$$\mathbf{m} = \frac{1}{2} \sum_{i=1}^n I_i \oint_{\Gamma_i} \mathbf{r}' \times d\mathbf{r}', \quad \mathbf{m} = \frac{1}{2} \int \mathbf{r}' \times \mathbf{J}(\mathbf{r}') d^3r'. \quad (5.82)$$

- Dipole magnetic field

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \left[ \frac{3\mathbf{r}(\mathbf{m} \cdot \mathbf{r}) - r^2\mathbf{m}}{r^5} \right]. \quad (5.83)$$

- Torque on dipole  $\mathbf{N} = \mathbf{m} \times \mathbf{B}$ .

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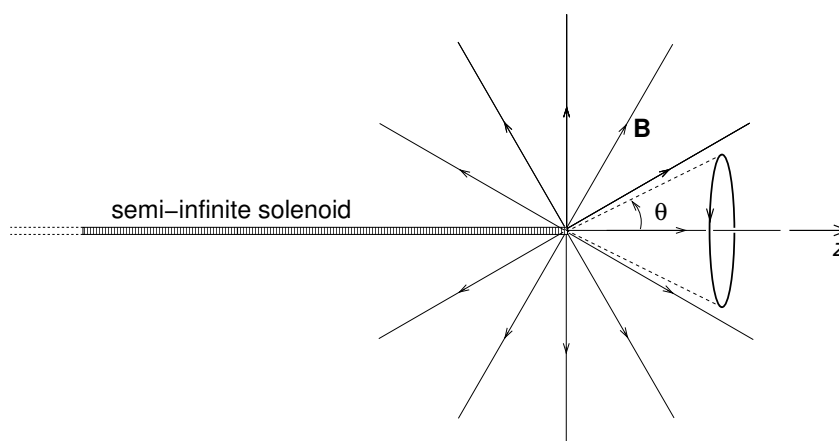
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## Exercises on Chapter 5

- 5–1 Two parallel wires are separated by distance  $a$  and carry currents  $I_1$  and  $I_2$  in the same direction. Find the force per unit length of wire. Include a diagram showing the direction of the force. If  $I_1 = I_2 = 1$  A and  $a = 1$  m, what is the magnitude of the force per unit length?
- 5–2 Using the equation for the vector potential in terms of the current density (Eq. 5.21) find the vector potential of an infinite straight wire along the  $z$  axis carrying current  $I$ .
- 5–3 We can add the gradient of a scalar field  $U(\mathbf{r})$  to  $\mathbf{A}(\mathbf{r})$  without changing  $\mathbf{B}(\mathbf{r})$ . This is called a gauge transformation. The “gauge” of the vector potential is determined by the value of  $\nabla \cdot \mathbf{A}$ . Show that in magnetostatics  $\nabla \cdot \mathbf{A} = \nabla^2 U(\mathbf{r})$ .
- 5–4 (a) Find the vector potential of the constant magnetic field  $\mathbf{B}_0 = (B_x^0, B_y^0, B_z^0)$ , (b) check that the vector potential you find does give the desired magnetic field, (c) find  $\nabla \cdot \mathbf{A}$  and check it is what is expected in magnetostatics. [Hint: first find the vector potential  $\mathbf{A}^{(z)}(\mathbf{r})$  of the simpler constant field  $B_z^0 \hat{\mathbf{z}}$  by writing down the components of  $\nabla \times \mathbf{A}^{(z)}$  in Cartesian coordinates before appealing to the symmetry of the problem, and then integrating.]
- 5–5 A steady current  $I$  flows down a long cylindrical wire of radius  $b$ . Find the magnetic field both inside and outside the wire.
- 5–6 A semi-infinite solenoid of radius  $a$ , has  $n$  turns per unit length, extends from  $z = -\infty$  to  $z = 0$  along the  $z$  axis and carries current  $I$  in the  $+\hat{\phi}$  direction. Magnetic flux is confined to the solenoid, but emerges isotropically from its end at the origin as shown below.



- (a) On the cone of half-angle  $\theta$  with apex at the origin there is a circular loop (as shown) with all points on the loop being at distance  $r \gg a$  from the origin. Find the magnetic flux passing through this loop. (b) Find the magnetic vector potential at the point  $(r, \theta, \phi)$ ,

and take its curl to find the magnetic field. [The expression for magnetic flux through a loop in terms of the vector potential may be useful here.]

5–7 By taking the curl of the vector potential for a magnetic dipole with moment  $\mathbf{m}$  located at the origin, find its magnetic field using index notation.

5–8 A circular current loop in the  $xy$  plane has radius  $a$  and is centred on the origin. It carries current  $I$  in the  $\phi$ -direction. There is a uniform magnetic field  $\mathbf{B}(\mathbf{r}) = B_0(\cos\theta\hat{\mathbf{z}} + \sin\theta\hat{\mathbf{y}})$  present. By integrating the torque  $d\mathbf{N} = \mathbf{r}' \times d\mathbf{F}$  on line element  $d\mathbf{r}'$  of the current loop at  $\mathbf{r}'$ , find the torque on the entire current loop. Compare your result with the result you would get by first finding the current loop's dipole moment, and then applying the formula for the torque on a magnetic dipole in a magnetic field.

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## 6 Magnetism of materials

### Learning objectives

- To learn that a material placed in a magnetic field acquires a magnetic dipole moment per unit volume described by the magnetisation field, and that magnetisation results in surface and volume magnetisation currents.
- To be able to calculate the magnetisation currents, magnetic field and the magnetic dipole moment of a permanently magnetised object.
- To understand that the magnetisation currents must be added to any free currents present when applying Ampère's law, and that this leads to the introduction of a new field  $\mathbf{H}$ .
- To be able to apply the boundary conditions on  $\mathbf{B}$  and  $\mathbf{H}$  at an interface between two materials.
- To learn that for many materials the magnetisation is proportional to  $\mathbf{H}$ , with the constant of proportionality being the magnetic susceptibility, which in turn is related to the permeability.
- To understand the causes of paramagnetism, and how a semi-classical treatment leads to Curie's law.
- To understand the causes of ferromagnetism, and that it requires no applied magnetic field to sustain it.
- To know how the magnetisation of a ferromagnetic material changes with applied magnetic field or  $\mathbf{H}$ , and to be able to calculate the work done as a ferromagnetic material is brought around a hysteresis loop.
- To understand the concept of magnetic circuits, and to be able to calculate magnetic fields in simple magnetic circuits.

### 6.1 Magnetic materials

While in an external magnetic field, a *paramagnetic* substance becomes magnetised with a net dipole moment parallel to the magnetic field. Paramagnetism occurs if, for a particular substance, the atomic electrons have unpaired spin allowing the electron magnetic moments



of those unpaired electrons to experience a torque aligning them with  $\mathbf{B}$ . *Ferromagnetism* is due to the quantum mechanical exchange interaction causing long-range ordering of electron spins within a “domain” — a concentration of aligned magnetic dipoles. If a material is neither paramagnetic nor ferromagnetic, it will show (usually weak) *diamagnetism*. This is due to the effect of the magnetic field acting on orbiting electrons and changing the magnetic moment associated with their orbital motion in such a way that the net dipole moment becomes anti-parallel with  $\mathbf{B}$ . Diamagnetism occurs for all atoms, but in paramagnetic and ferromagnetic materials is masked by the typically stronger paramagnetism and ferromagnetism.

## 6.2 Magnetisation field

In the presence of a magnetic field matter becomes *magnetised* in an analogous way to that in which matter becomes polarised in the presence of an electric field. Magnetic dipole moments due to electron spin or orbital motion get partially or fully lined up in some direction giving the material a net magnetic dipole moment. To describe this we define the *magnetisation* field  $\mathbf{M}(\mathbf{r})$  in a similar way to the polarisation field in electrostatics, namely the magnetisation field is defined as the magnetic dipole moment per unit volume. The magnetisation field  $\mathbf{M}$  has units of magnetic dipole moment ( $\text{A m}^2$ ) divided by those of volume ( $\text{m}^3$ ), i.e. its units are ( $\text{A m}^{-1}$ ). In the same way that in electrostatics the electric field of a polarised object is identical to that due to its polarisation charges, the magnetic field of a magnetised object is identical to that of its *magnetisation currents*, also known as “bound currents”.

## 6.3 Magnetisation currents and Ampère's law

The vector potentials of a magnetic dipole with moment  $\mathbf{m}$  at the origin, of a magnetic dipole with moment  $\mathbf{m}$  located at  $\mathbf{r}'$ , and of a magnetised object are given by

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}, \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{R}}{R^3}, \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{M}(\mathbf{r}') \times \mathbf{R}}{R^3} d^3r' \quad (6.1)$$

where, as usual,  $\mathbf{R} = \mathbf{r} - \mathbf{r}'$ .

To derive the magnetisation currents of a magnetised object we will need the following: an identity for  $\nabla' R^{-1}$ , the product rule for  $\nabla \times (a\mathbf{F})$ , and a corollary to Gauss' Theorem, all of which are obtained in Exercise 6–1. The derivation of magnetisation currents proceeds as

follows. First re-write the cross product in the integral in Eq. 6.1

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \mathbf{M}(\mathbf{r}') \times \frac{\hat{\mathbf{R}}}{R^2} d^3r'. \quad (6.2)$$

Then, use  $\nabla' R^{-1} = +R^{-2}\hat{\mathbf{R}}$  (Eq. B.9) to find

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \mathbf{M}(\mathbf{r}') \times \nabla' \left( \frac{1}{R} \right) d^3r'. \quad (6.3)$$

Next, using the rule for the curl of the product of scalar and vector fields (Eq. I.9) we obtain

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\nabla' \times \mathbf{M}}{R} d^3r' - \frac{\mu_0}{4\pi} \int_V \nabla' \times \left( \frac{\mathbf{M}}{R} \right) d^3r'. \quad (6.4)$$

Finally, using the corollary to Gauss' theorem  $\int_V \nabla \times \mathbf{F} d^3r = - \oint_S \mathbf{F} \times d\mathbf{S}$  we get

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\nabla' \times \mathbf{M}}{R} d^3r' + \frac{\mu_0}{4\pi} \oint_S \left( \frac{\mathbf{M}}{R} \right) \times d\mathbf{S}', \quad (6.5)$$

$$\therefore \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\nabla' \times \mathbf{M}}{R} d^3r' + \frac{\mu_0}{4\pi} \oint_S \frac{\mathbf{M} \times d\mathbf{S}'}{R}. \quad (6.6)$$

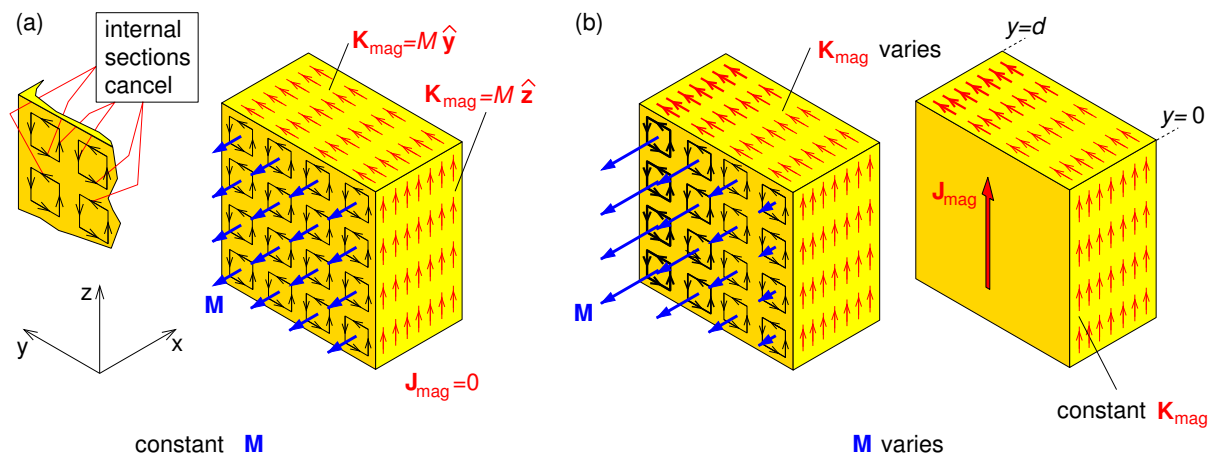
Hence, the field of a magnetised object is the same as would be produced by “magnetisation currents”

$$\mathbf{J}_{\text{mag}}(\mathbf{r}) = \nabla \times \mathbf{M}(\mathbf{r}) \quad (\text{volume}), \quad \mathbf{K}_{\text{mag}}(\mathbf{r}) = \mathbf{M}(\mathbf{r}) \times \hat{\mathbf{n}} \quad (\text{surface}). \quad (6.7)$$

To visualise these magnetisation currents, consider a rectangular block of material uniformly magnetised in the  $-\hat{\mathbf{x}}$  direction as in Fig. 6.1(a). We can imagine the magnetisation field as being due to the tiny current loops shown. If  $\mathbf{M}$  is constant throughout the material then each of these fictitious current loops carries an identical current, such that inside the material the current on one side of a loop is in the opposite direction to the current on the nearest side of an adjacent loop and cancels it out (see inset in Fig. 6.1a). So,  $\mathbf{J}_{\text{mag}}(\mathbf{r}) = 0$  inside the uniformly magnetised material, as it must be since  $\nabla \times \mathbf{M} = 0$ .

There is no magnetisation surface current density in the surfaces perpendicular to  $\mathbf{M}$  (with darker shading), i.e. parallel to the  $yz$  plane in Fig. 6.1(a), because  $\mathbf{M} \times \hat{\mathbf{n}} = 0$  there. However,

for the loops which have one side next to the surface, the currents on that side of the loops have no opposing currents to cancel with. The result is a magnetisation surface current  $\mathbf{K}_{\text{mag}}(\mathbf{r}) = \mathbf{M}(\mathbf{r}) \times \hat{\mathbf{n}}$  such that on the surface of the side with lighter shading parallel to the  $xz$  plane  $\mathbf{K}_{\text{mag}}(\mathbf{r}) = M\hat{\mathbf{z}}$  whereas on the top surface with lighter shading (parallel to the  $xy$  plane)  $\mathbf{K}_{\text{mag}}(\mathbf{r}) = M\hat{\mathbf{y}}$ .



**Figure 6.1:** Physical interpretation of magnetisation currents. (a) Uniform magnetisation. Inset shows detail of cancellation of internal currents (b)  $M_x$  varies with  $y$  showing: left – internal currents don't cancel; right – the relative magnitudes of the surface magnetisation current densities  $\mathbf{K}_{\text{mag}}$  as it varies over the surfaces, as well as the direction of the volume magnetisation current density  $\mathbf{J}_{\text{mag}}$ .



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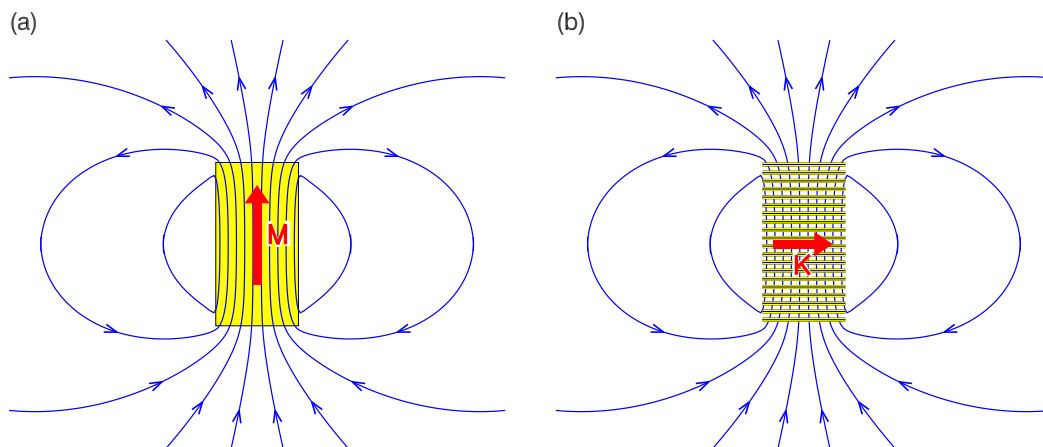
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If  $\mathbf{M}$  varies with position inside the material, as well as the surface magnetisation currents there will also be a volume magnetisation current density provided  $\nabla \times \mathbf{M}(\mathbf{r}) \neq 0$ . One example is shown in Fig. 6.1(b) where  $\mathbf{M}(\mathbf{r})$  depends only on  $y$  and varies linearly with  $y$ , i.e.  $\mathbf{M}(\mathbf{r}) = M_x(y) \hat{\mathbf{x}} = -(M_0 + ay) \hat{\mathbf{x}}$  such that  $\mathbf{J}(\mathbf{r}) = \nabla \times \mathbf{M}(\mathbf{r}) = -dM_x/dy \hat{\mathbf{z}} = a \hat{\mathbf{z}}$ . The resulting surface and volume current densities are illustrated in Fig. 6.1(c) where on the top surface  $\mathbf{K}_{\text{mag}}(\mathbf{r}) = M_x(y) \hat{\mathbf{y}}$  which varies with position, while on the sides parallel to the  $x$ - $z$  plane the magnetisation surface current densities are constant with the values  $\mathbf{K}_{\text{mag}}(x, 0, z) = -M_x(0) \hat{\mathbf{z}} = M_0 \hat{\mathbf{z}}$  and  $\mathbf{K}_{\text{mag}}(x, d, z) = M_x(d) \hat{\mathbf{z}} = -(M_0 + ad) \hat{\mathbf{z}}$ .

### 6.3.1 Cylindrical magnets

As an example of magnetisation currents, we consider a permanent magnet in the form of a short cylinder with uniform magnetisation  $\mathbf{M}$  along its axis as in Fig. 6.2(a). The magnetisation surface current  $\mathbf{K}_{\text{mag}}(\mathbf{r}) = \mathbf{M}(\mathbf{r}) \times \hat{\mathbf{n}}$  is zero on the two ends, but on the cylindrical surface  $\mathbf{K}_{\text{mag}}(\mathbf{r}) = M \hat{\boldsymbol{\phi}}$  (Fig. 6.2b). This surface magnetisation current distribution would give an identical magnetic field to a free surface current density  $\mathbf{K} = nI \hat{\boldsymbol{\phi}}$  where  $I = M/n$  is the current in a tightly-wound solenoid of the same size as the magnet (Fig. 6.2b), and  $n$  is the number of turns per metre.



**Figure 6.2:** Magnetic field of (a) a short cylinder with uniform magnetisation  $\mathbf{M} = M\hat{\mathbf{z}}$  parallel to its axis, and (b) a solenoid of identical size with  $n$  coils per unit length and carrying current  $I = M/n$ , effectively a surface current  $\mathbf{K} = nI\hat{\boldsymbol{\phi}}$ .

## 6.4 Ampère's law in magnetised materials

For magnetic materials we must include  $\mathbf{J}_{\text{mag}}$  as well as the “free current”  $\mathbf{J}_f$ , i.e.  $\mathbf{J}$  used previously must be replaced by the total current density as free currents and magnetisation currents are equally good at producing magnetic fields. Hence,

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J}_f + \mathbf{J}_{\text{mag}}) = \mu_0(\mathbf{J}_f + \nabla \times \mathbf{M}). \quad (6.8)$$

Hence, Ampère's law in magnetised materials is

$$\nabla \times \mathbf{H} = \mathbf{J}_f, \quad \text{where} \quad \mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}. \quad (6.9)$$

$\mathbf{H}$  is sometimes called *magnetic intensity* but more often simply “ $\mathbf{H}$ ”. It has units  $\text{A m}^{-1}$ . Using Stokes' theorem we obtain Ampère's law in integral form

$$\oint_{\Gamma} \mathbf{H} \cdot d\mathbf{r} = I_{f,\text{encl}}, \quad (6.10)$$

where  $I_{f,\text{encl}}$  is the free current enclosed by loop  $\Gamma$ .

### 6.4.1 Boundary Conditions on $\mathbf{B}$ and $\mathbf{H}$

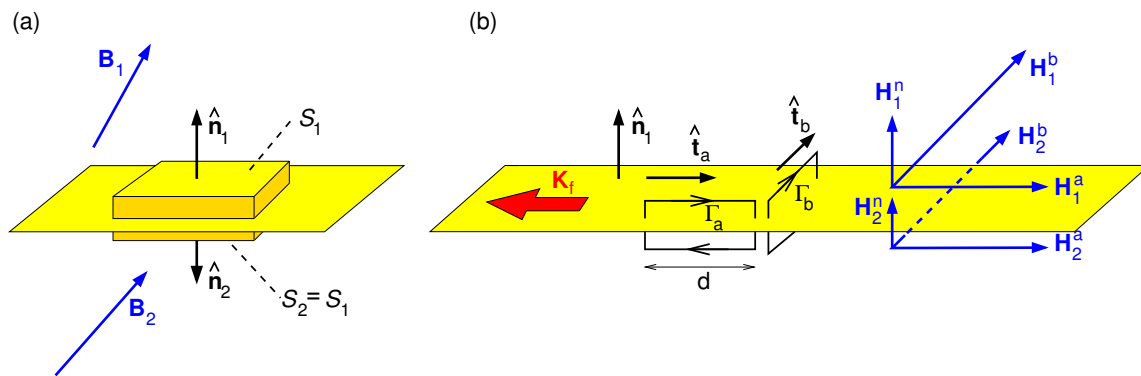
We obtain these boundary conditions from the no-magnetic-charge law and Ampère's law in integral form. Consider the Gaussian pillbox shown in Fig. 6.3(a) but with infinitesimally narrow sides. The surface integral of  $\mathbf{B}$  over the pillbox with identical top and bottom surface areas  $S_1 = S_2$  is given by

$$\oint_S \mathbf{B}(\mathbf{r}) \cdot d\mathbf{S} = 0, \quad (\mathbf{B}_1 \cdot \hat{\mathbf{n}}_1 S_1 + \mathbf{B}_2 \cdot \hat{\mathbf{n}}_2 S_2) = 0. \quad (6.11)$$

$$\therefore \mathbf{B}_{\text{above}}^{\perp} = \mathbf{B}_{\text{below}}^{\perp}, \quad (6.12)$$

and we see that the component of  $\mathbf{B}$  normal to the interface is unchanged across the boundary.

To find the boundary condition on  $\mathbf{H}$  we apply Ampère's law  $\oint \mathbf{H} \cdot d\mathbf{r} = I_{f,\text{encl}}$  to Ampèrian loops  $\Gamma_a$  and  $\Gamma_b$  of length  $d$  and infinitesimal width (Fig. 6.3b). Consider loop  $\Gamma_a$  whose sides



**Figure 6.3:** Geometry for obtaining the boundary conditions on (a) the normal component of  $\mathbf{B}$ , and (b) the component of  $\mathbf{H}$  parallel to the interface between material 1 (above the interface) and material 2 (below). The thickness of the Gaussian pill box and the widths of the Amperian loops are exaggerated in the diagrams, and should be considered as infinitesimal.

are parallel to  $\mathbf{K}_f$  such that enclosed current is zero, and loop  $\Gamma_b$  whose sides are perpendicular to  $\mathbf{K}_f$  and has enclosed current  $K_f d$ ,

$$\oint_{\Gamma_a} \mathbf{H} \cdot d\mathbf{r} = (\mathbf{H}_1 \cdot \hat{\mathbf{t}}_a d - \mathbf{H}_2 \cdot \hat{\mathbf{t}}_a d) = 0, \quad (6.13)$$

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$$\oint_{\Gamma_b} \mathbf{H} \cdot d\mathbf{r} = (\mathbf{H}_1 \cdot \hat{\mathbf{t}}_b d - \mathbf{H}_2 \cdot \hat{\mathbf{t}}_b d) = K_f d. \quad (6.14)$$

So the component of  $\mathbf{H}$  parallel to both the surface and  $\mathbf{K}_f$  is unchanged, and the component of  $\mathbf{H}$  which is both parallel to the surface and perpendicular to  $\mathbf{K}_f$  is changed by an amount  $K_f$ . Combining the results for the two parallel components of  $\mathbf{H}$  we obtain

$$(\mathbf{H}_{\text{above}}^{\parallel} - \mathbf{H}_{\text{below}}^{\parallel}) = \mathbf{K}_f \times \hat{\mathbf{n}}. \quad (6.15)$$

## 6.5 Magnetic susceptibility and permeability

For linear materials,  $\mathbf{M} = \chi_M \mathbf{H}$  where  $\chi_M$  is the *magnetic volume susceptibility*, and so

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_M)\mathbf{H} = \mu\mathbf{H} \quad (6.16)$$

where  $\mu$  is the *permeability* of the material, and  $\mu_0$  is the permeability of free space. Since the magnetic dipoles are at the atomic/molecular level, with  $\mathbf{M}$  depending on the number density of atoms/molecules (which varies by orders of magnitude between gases and solids), it makes sense when comparing the magnetic properties of materials to divide the magnetic volume susceptibility by the density. This quantity, called the *magnetic mass susceptibility* is tabulated in Table 6.1 for several elements, compounds and substances.

Substance	$\chi_M/\rho$	Substance	$\chi_M/\rho$	Substance	$\chi_M/\rho$	Substance	$\chi_M/\rho$
Aluminium	+0.82	Bismuth	-1.70	Copper	-0.107	Gallium	-0.30
Germanium	-0.15	Gold	-0.19	Helium	-0.59	Hydrogen	-2.49
Krypton	-0.41	Lead	-0.15	Mercury	-0.21	Neon	-0.41
Nitrogen	-0.54	Oxygen	+133.6	Phosphorus	-1.13	Platinum	+1.22
Potassium	+0.65	Silicon	-0.16	Silver	-0.25	Sodium	+0.75
Sulphur	-0.62	Uranium	+2.19	CO <sub>2</sub>	-0.59	H <sub>2</sub> O	-0.90
HCl	-0.75	H <sub>2</sub> SO <sub>4</sub>	-0.50	NaCl	-0.64	NH <sub>3</sub>	-1.38
NO	+59.3	Perspex	-0.5	Polyethylene	+0.2	P.V.C.	-0.75

**Table 6.1:** Mass susceptibilities,  $\chi_M/\rho$  ( $10^{-8} \text{ kg}^{-1} \text{ m}^3$ ) at 20° C of some common elements, compounds and materials (data taken from Kaye & Laby [http://www.kayelaby.npl.co.uk/general\\_physics/2\\_6/2\\_6\\_6.html](http://www.kayelaby.npl.co.uk/general_physics/2_6/2_6_6.html)).

## 6.6 Paramagnetism

Pierre Curie discovered that for ordinary temperatures and magnetic fields paramagnetic materials obey  $\chi_M \equiv (M/H) \propto 1/T$ . Paramagnetism is analogous to orientational polarisability of permanent molecular electric dipoles. Magnetic torques on atomic dipole moments cause paramagnetism, and it is the magnetic moments associated with unpaired spin that are responsible. Unpaired electron spin will occur where there is an unfilled atomic sub-shell. However, in metals only those electrons with energies near the Fermi energy can respond in this way and so paramagnetism in metals is weak and does not follow the Curie law.

For a particle of mass  $M_p$  and charge  $q$  in a circular orbit of radius  $R$  with speed  $v$  (Fig. 5.4) the magnitudes of the magnetic dipole moment  $\mathbf{m}_{\text{dip}}$  and angular momentum  $\mathbf{L}$  are

$$m_{\text{dip}} = \frac{1}{2}qvR, \quad \text{and} \quad L = M_p v R, \quad (6.17)$$

where the dipole moment is the product of the current ( $q$  divided by orbital period  $2\pi R/v$ ) and the area of the orbit ( $\pi R^2$ ). The *gyromagnetic ratio* is defined as

$$\frac{m_{\text{dip}}}{L} = \frac{q}{2M_p}, \quad \text{and} \quad m_{\text{dip}} = \frac{qL}{2M_p}. \quad (6.18)$$

For atoms we need to take account of quantum mechanics, in which the quantum of angular momentum is  $\hbar \equiv h/2\pi$  where  $h$  is Planck's constant. The natural unit of  $m_{\text{dip}}$  is therefore the “Bohr magneton”

$$\mu_B = \frac{e \hbar}{2m_e}. \quad (6.19)$$

The total angular momentum  $\mathbf{J} = \mathbf{L} + \mathbf{S}$  is the sum of the orbital angular momentum of electrons and their spin. For  $\mathbf{B} = B \hat{\mathbf{z}}$  the quantum number  $M$  for the  $z$ -component of  $\mathbf{J}$  is relevant here, such that the  $z$ -component of the magnetic dipole moment  $\mathbf{m}_{\text{dip}}$  is

$$m_{\text{dip},z} = -g_J \mu_B M, \quad g_J = 1 + \frac{J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}, \quad (6.20)$$

where  $g_J$  is the “Landé g-factor”. Note that if  $L = 0$  (no orbital angular momentum) then  $J = S$  and  $g_J = 2$  for electron spin, and that if  $S = 0$  (no spin angular momentum) then  $J = L$  and  $g_J = 1$  for electron orbital angular momentum.



As an aside, the Zeeman effect is the atomic energy-level splitting in a magnetic field

$$\Delta E = -\mathbf{m}_{\text{dip}} \cdot \mathbf{B} = +g_J \mu_B M B \quad (6.21)$$

and this provides an important method for radio astronomers to determine the magnetic field present in some astrophysical sources.

### 6.6.1 Derivation of susceptibility

Paramagnetism is due to partial alignment of permanent atomic magnetic dipoles. For a full treatment, a quantum statistical mechanical approach is required, taking account of the quantum numbers for total angular momentum  $\mathbf{J}$ , orbital angular momentum  $\mathbf{L}$ , and spin  $\mathbf{S}$ . However, the problem is simplified for the case atoms with zero orbital angular momentum, as we then have  $g_J = 2$  and only two possible spin states  $M_s = \pm \frac{1}{2}$ . The corresponding magnetic dipole moments are

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$$m_{\text{dip},z} = -gJ\mu_B M_s = -2\mu_B \left(\pm \frac{1}{2}\right) = \mp \mu_B, \quad (6.22)$$

and the corresponding potential energy values are

$$W = -\mathbf{m} \cdot \mathbf{B} = -m_{\text{dip},z} B = \pm \mu_B B. \quad (6.23)$$

Using the Boltzmann distribution, for temperature  $T$  the probabilities of the electron magnetic moment being parallel and antiparallel to the magnetic field are

$$P_{\text{parallel}} = P_0 e^{\mu_B B/kT}, \quad P_{\text{antiparallel}} = P_0 e^{-\mu_B B/kT} \quad (6.24)$$

where  $P_0$  is a constant and  $(P_{\text{parallel}} + P_{\text{antiparallel}}) = 1$ . Then, for number density  $N$  of unpaired electrons the magnetisation is

$$\mathbf{M} = (P_{\text{parallel}} - P_{\text{antiparallel}}) N \mu_B \hat{\mathbf{B}} = \frac{e^{\mu_B B/kT} - e^{-\mu_B B/kT}}{e^{\mu_B B/kT} + e^{-\mu_B B/kT}} N \mu_B \hat{\mathbf{B}}. \quad (6.25)$$

For  $\mu_B B \ll kT$ , i.e. for ordinary temperatures and magnetic fields,

$$\mathbf{M} \approx \frac{\mu_B B}{kT} N \mu_B \hat{\mathbf{B}} = \frac{\mu_B}{kT} N \mu_B (\mu_0 \mathbf{H}), \quad (6.26)$$

and the magnetic susceptibility becomes

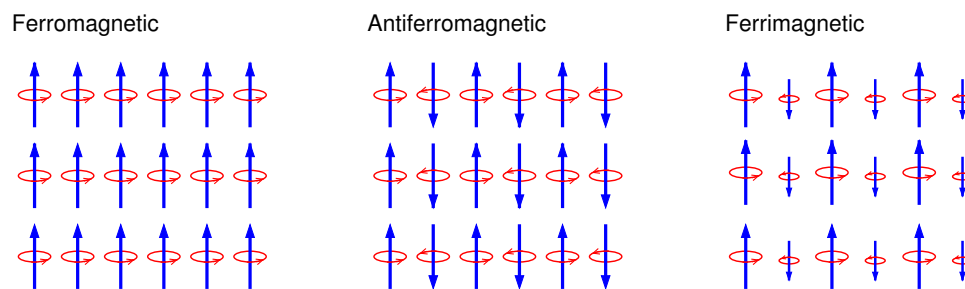
$$\chi_m \approx \frac{N \mu_0 \mu_B^2}{kT}, \quad (6.27)$$

consistent with the Curie law.

## 6.7 Ferromagnetism

Ferromagnetism is *non-linear*, and no external fields are needed to keep a ferromagnet magnetised. Ferromagnetism is due to long-range ordering of unpaired electron spins causing them to line up parallel to each other. Anti-ferromagnetism is similar, except that neighbouring dipoles point in opposite directions. Ferrimagnetism has oppositely pointing neighbouring dipoles of

different magnitude which don't cancel. These three possibilities are illustrated schematically in Fig. 6.4.

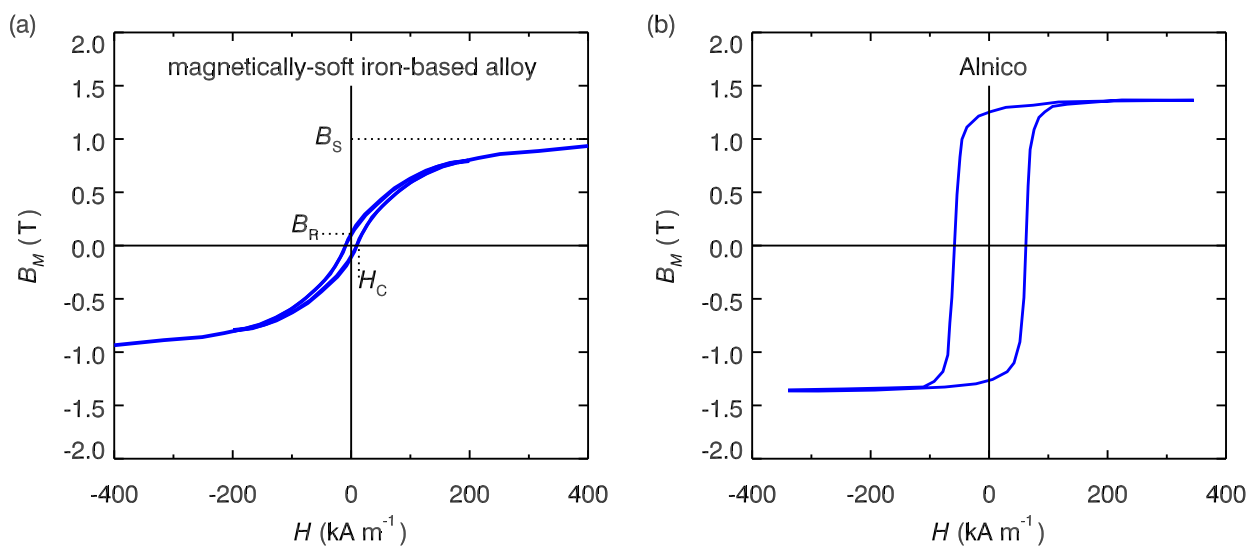


**Figure 6.4:** Arrangement of dipoles in crystal lattice planes of ferromagnetic, antiferromagnetic and ferrimagnetic materials.

Alignment occurs in microscopic “domains” with the magnetisation direction of each domain usually being random, and giving an overall unmagnetised state. When placed in an external magnetic field, domains with magnetisation aligned partially with  $\mathbf{B}$  grow at the expense of less favourably aligned ones so that the average magnetisation is in the direction of  $\mathbf{B}$ . Torques  $\mathbf{N} = \mathbf{m} \times \mathbf{B}$  try to align individual dipoles with  $\mathbf{B}$ , but they like to stay parallel to their neighbours, and a lot of work is needed to shift the domain boundaries to achieve greater alignment. For ferromagnetic materials, the magnetic field  $\mathbf{B}_M = \mu_0 \mathbf{M}$  due to the resulting magnetisation  $\mathbf{M}$  is typically much larger than the applied field  $\mathbf{B}_0 = \mu_0 \mathbf{H}$ . In ferromagnetism and ferrimagnetism the material becomes paramagnetic above a critical temperature at which a phase transition occurs called the *Curie temperature* (after Pierre Curie) and in anti-ferromagnetism above the *Néel temperature* (after French physicist Louis Néel 1904–2000). Practical applications of this phase transition occur in electro-optical devices and hard disk drives when heat is used to erase and write data.

Typically one would examine the magnetic properties of a ferromagnetic material by placing the sample inside a solenoid and varying the current (and consequently  $\mathbf{H}$ ) sinusoidally, often at mains frequency (50 Hz or 60 Hz), and displaying  $B_M$  vs.  $H$  on an oscilloscope or computer screen. An example is shown in Fig. 6.5(a) for a magnetically-soft iron-based amorphous alloy. What is found is that  $B_M$  remains positive even when  $H$  returns to zero following its maximum positive value, and that  $B_M$  reaches zero only after a time lag when  $H$  is sufficiently negative. Provided  $H_{\max}$  is not too large  $B_M$  vs.  $H$  is approximately linear. Such a material would be used mainly for magnetic cores in transformers. For larger applied fields,  $\mathbf{M}$  saturates when all domains having spins partially aligned with  $\mathbf{H}$  have grown as much as is possible.

This type of non-reversible process is called *hysteresis*. To bring the sample through one cycle of the “hysteresis loop” work must be done per unit volume of the ferromagnetic material equal



**Figure 6.5:** Hysteresis in (a) a magnetically-soft iron-based amorphous alloy, (b) a magnetically-hard alloy of iron, aluminium, nickel and cobalt, Alnico.

to the loop's area,  $W = \oint H dB$ . Depending on the practical application of the magnetic material, which can vary enormously from computer discs to permanent magnets to magnetic films for fridge magnets to magnetic shielding of electronics, one would choose a material which

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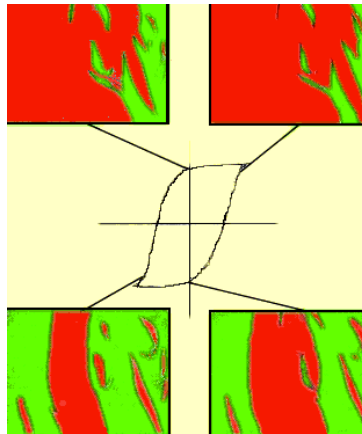


has “low hysteresis” (area of hysteresis loop is small), and is termed “magnetically soft” (good for magnetic cores for transformers and for electromagnets), or one which has “high hysteresis” and is termed “magnetically hard” (good for permanent magnets), or one which has the right shape of the hysteresis loop for the job in question.

The hysteresis loop of a magnetic material is characterised by its *remanence*  $B_R$  (T), its *coercivity*  $H_C$  ( $\text{A m}^{-1}$ ) and its *saturation*  $B_S$  (T), and these three quantities are labelled in Fig. 6.5(a). For this particular amorphous alloy:  $B_R = 0.1$  T,  $H_C = 10$   $\text{A m}^{-1}$  and  $B_S = 1$  T. Again, depending on the application, the shape of the hysteresis loop is important – it depends not only on the material but also on whether it has been annealed, i.e. heated and cooled in the presence of a magnetic field aligned in a particular direction. An example of a hysteresis loop with a very different shape to that of the magnetically-soft amorphous alloy is shown in Fig. 6.5(b), in this case for a magnetically hard alloy of iron, aluminium, nickel and cobalt “Alnico”, which would be used for permanent magnets.

A recent development allows us to image ferromagnetic domains using scanning electron microscopes (SEM). These microscopes work by having a beam of keV to multi-keV energy electrons scan across the surface of the electrically conducting sample in a raster scan pattern. If the object is non-conducting it must be coated with a thin layer of a good conductor such as gold. Some of the scanning electrons are absorbed and the resulting current depends on the local surface height and material properties, and is used to construct an image of the surface which is displayed on a computer screen. Electrons scattered off the surface can also be collected in real-time, as can X-rays emitted as a result of the surface bombardment, and are also used in construction of the image in some SEMs.

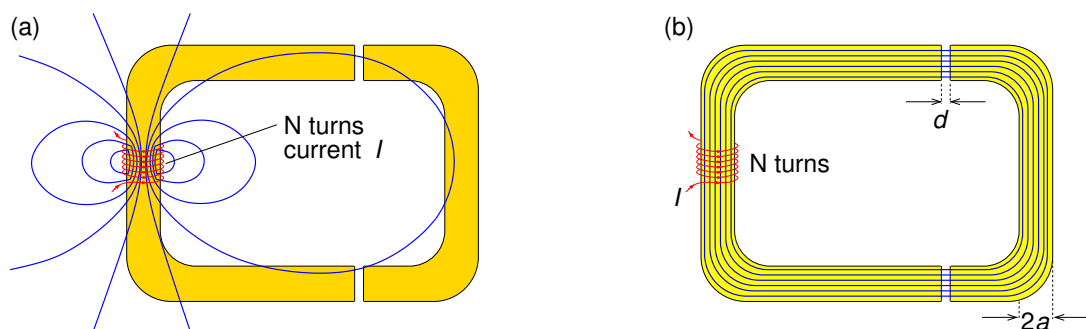
Recently, a detector was developed to collect and analyse the spin state of the scattered electrons, which in turn depends on the magnetisation direction of the patch of surface being imaged. This scanning electron microscope with polarisation analysis (SEMPA) technique enables images of the domain structure on the surface of a ferromagnetic material to be made. Figure 6.6 shows images of magnetic domains and domain wall dynamics in Metglas<sup>®</sup> ribbons. SEMPA measurements of the magnetic domains are shown at four points on the hysteresis loop (centre) showing the magnetic state corresponding to each image.



**Figure 6.6:** Hysteresis in an amorphous alloy. Red (darker shading) indicates positive magnetisation, green (lighter shading) indicates negative. The small, relatively immobile domains are pinned by stress fields induced during fabrication. Each of the images is approximately 980 microns wide. Image is courtesy of A. Gavrin et al., from [http://www.nist.gov/cnst/epg/wall\\_motion\\_proj.cfm](http://www.nist.gov/cnst/epg/wall_motion_proj.cfm) (reproduced with permission).

### 6.7.1 Magnetic circuits

Consider two rods made of plastic, each of which is heated in order to bend it into a horseshoe shape. The two plastic horseshoes are placed end-to-end almost touching but leaving two short air gaps between them. Wires are wrapped around part of one of the horseshoes to form a short solenoid as shown in Fig. 6.7(a). A current passing through the solenoid will produce a magnetic field which, because the magnetic susceptibility of plastic is tiny, will be only slightly different to the field the solenoid would produce in air.



**Figure 6.7:** (a) Two horseshoes made of plastic and placed end-to-end almost touching but leaving small air gaps, and having a short solenoid wound around part of one horseshoe. (b) As part (a) but with the horseshoes being made of ferrite core material.

Now consider horseshoes of identical size and shape to the plastic horseshoes, but made of ferrite core material. Ferrite is magnetically soft and has  $\mu \gg \mu_0$ . There is an identical short solenoid with  $N$  coils carrying current  $I$  as shown in Fig. 6.7(b). The magnetic field in the

ferrite core will be dramatically different to that in the plastic horseshoes as can be shown using the boundary condition on  $\mathbf{H}$  at the boundary between the ferrite core material and the air. Since there are no free surface currents in the ferrite, the component of  $\mathbf{H}$  parallel to the surface,  $\mathbf{H}^{\parallel}$ , must be continuous across the boundary between ferrite and air

$$\mathbf{B}_{\text{ferrite}}^{\parallel} / \mu_{\text{ferrite}} = \mathbf{B}_{\text{air}}^{\parallel} / \mu_{\text{air}}. \quad (6.28)$$

Hence,  $\mathbf{B}_{\text{ferrite}}^{\parallel} \gg \mathbf{B}_{\text{air}}^{\parallel}$  as  $\mu_{\text{ferrite}} \gg \mu_0$ , and the magnetic field will be largely confined to the ferrite core, with the field lines making loops as shown. Using the boundary condition on the normal component of  $\mathbf{B}$ , the magnetic field in the two air gaps between the two horseshoes must be unchanged across the boundaries as the air gap surfaces are normal to  $\mathbf{B}$  inside the ferrite core.

As the ferrite core effectively channels the magnetic field, effectively we have a *magnetic circuit*. As an example, we shall calculate the magnetic field in the air gaps for a permeability of  $\mu_1 = 5,000\mu_0$  for the horseshoe on the left and  $\mu_2 = 3,000\mu_0$  for the horseshoe on the right – strictly speaking ferromagnetic materials are non-linear, but we shall assume these values apply to initial magnetisation before the field saturates. The other values we shall use in this example are  $I = 0.2$  A,  $N = 50$ , the lengths around the two horseshoes being  $L_1 = 30$  cm and  $L_2 = 20$

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cm,  $a \ll L_2$  and the air gaps having width  $d = 0.2$  mm ( $\ll a$ ). Applying Ampère's law to the closed curve following the centre of the cross-sections of the ferrite core horseshoes

$$\oint \mathbf{H} \cdot d\mathbf{r} = I_{\text{encl}}, \quad \therefore L_1 \frac{B}{\mu_1} + L_2 \frac{B}{\mu_2} + 2d \frac{B}{\mu_0} = NI. \quad (6.29)$$

Substituting the numbers we find  $B = 0.024$  T.

## Summary of Chapter 6

### Macroscopic theory of magnetism

- The magnetisation field  $\mathbf{M}$  is defined as the magnetic dipole moment per unit volume.
- Paramagnets have  $\mathbf{M}$  parallel to the applied field, diamagnets have  $\mathbf{M}$  anti-parallel.
- Magnetised materials have magnetisation currents

$$\mathbf{J}_{\text{mag}}(\mathbf{r}) = \nabla \times \mathbf{M}(\mathbf{r}) \quad (\text{volume}), \quad \mathbf{K}_{\text{mag}}(\mathbf{r}) = \mathbf{M}(\mathbf{r}) \times \hat{\mathbf{n}} \quad (\text{surface}). \quad (6.30)$$

- Ampère's law in magnetised materials  $\nabla \times \mathbf{H} = \mathbf{J}_f$ ,  $\oint \mathbf{H} \cdot d\mathbf{r} = I_{f, \text{encl}}$ ,  
where  $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$ .
- Boundary Conditions between magnetic materials

$$\mathbf{B}_{\text{above}}^\perp = \mathbf{B}_{\text{below}}^\perp, \quad (\mathbf{H}_{\text{above}}^\parallel - \mathbf{H}_{\text{below}}^\parallel) = \mathbf{K}_f \times \hat{\mathbf{n}}. \quad (6.31)$$

- Linear materials: susceptibility  $\chi_M$  and permeability  $\mu$

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(1 + \chi_M)\mathbf{H} = \mu\mathbf{H}. \quad (6.32)$$

### Paramagnetism

- Magnetic torques on atomic dipole moments cause paramagnetism.
- At normal temperatures and applied fields partial alignment occurs giving

$$\chi_M \equiv \frac{\mu_0 M}{B} \approx \frac{\mu_0 n m^2}{3kT}. \quad (6.33)$$



## Ferromagnetism

- Ferromagnets are non-linear due to long-range ordering causing unpaired electron spins to line up parallel with each other.
- Alignment occurs in microscopic domains with the direction of  $\mathbf{M}$  in each domain being random, and giving overall unmagnetised state.
- When a magnetic field is applied, torques  $\mathbf{N} = \mathbf{m} \times \mathbf{B}$  try to align individual dipoles with the applied field.
- Work is done to shift the domain boundaries to cause overall magnetisation.
- This non-reversible process is called hysteresis – a plot of  $B_M$  vs. applied  $H$  shows the hysteresis loop.
- The area of the loop gives the work done to take the sample around one cycle of the loop.

## Exercises on Chapter 6

- 6–1 Derive the following which are needed to obtain the magnetisation currents of a magnetised object: (a) Identity for  $\nabla' R^{-1}$ ,

$$\nabla' R^{-1} = +R^{-2} \hat{\mathbf{R}}. \quad (6.34)$$

- (b) Product rule for  $\nabla \times (a\mathbf{F})$ ,

$$\nabla \times (a\mathbf{F}) = (\nabla a) \times \mathbf{F} + a \nabla \times \mathbf{F}. \quad (6.35)$$

- (c) Corollary to Gauss' Theorem,

$$\int_V \nabla \times \mathbf{F} \, d^3r = - \oint_S \mathbf{F} \times d\mathbf{S}. \quad (6.36)$$

- 6–2 A thin disc of magnetised material is coincident with the  $xy$  plane. It is of thickness  $s$  and radius  $a$  and has magnetisation  $\mathbf{M} = M_0 \hat{\mathbf{z}}$ . Find the magnetisation current, and from this find the magnetic dipole moment of the disc. Compare this with what you would get by multiplying the disc's volume by  $\mathbf{M}$ .

- 6–3 Consider a permanent magnet in the form of a short cylinder of radius  $a$  extending along the  $z$  axis from  $z = -L$  to  $z = +L$  and having uniform magnetization  $\mathbf{M} = M_0 \hat{\mathbf{z}}$ . (a) Find  $\mathbf{B}$  and  $\mathbf{H}$  at all points  $(0, 0, z)$  on the cylinder's axis, and plot  $\mathbf{B}(0, 0, z)$  and  $\mathbf{H}(0, 0, z)$  vs.  $z$ . (b) Discuss whether the result obtained in part (a) obeys Ampère's law for  $\mathbf{H}$  in integral form.
- 6–4 A cylindrical rod of radius  $a$  and length  $h \gg a$  is permanently magnetised along its length which coincides with the  $z$  direction, i.e.  $\mathbf{M} = M_0 \hat{\mathbf{z}}$ . (a) Find the surface magnetisation current  $\mathbf{K}_{\text{mag}}(\mathbf{r})$ , and use it together with Ampere's law to find  $\mathbf{B}$  and  $\mathbf{H}$  inside and outside the rod (assume  $h \rightarrow \infty$ ). (b) The rod is now bent into a circle of circumference  $(h + 2L)$  such that there is an air gap of width  $2L < a$ . Plot  $\mathbf{B}$  and  $\mathbf{H}$  along the axis of the magnet in the vicinity of the air gap for the case of  $a = 0.5$  and  $L = 0.2$ .
- 6–5 Consider a permanently magnetised sphere of radius  $a$  with uniform magnetisation  $\mathbf{M}(\mathbf{r}) = M_0 \hat{\mathbf{z}}$ . (a) Find the surface magnetisation current density, and use this to find the magnetic dipole moment of the sphere. Compare this with what you expect given the volume of the sphere and the magnetisation field. (b) Find  $\mathbf{B}$  and  $\mathbf{H}$  at the centre of the sphere.
- 6–6 Consider the hysteresis loops of the magnetically-soft iron-based amorphous alloy and the magnetically-hard alloy of iron, aluminium, nickel and cobalt shown in Fig. 6.5. (a) Estimate the work done to bring  $1 \text{ cm}^3$  of each material through one cycle of the hysteresis loop. (b) Two transformers operating at 50 Hz have magnetic cores of volume  $100 \text{ cm}^3$  (one of each type of material) and are (unwisely) operated at a current at which saturation occurs. How much power is lost as heat in each case? [You could print Fig. 6.5 and estimate the area by drawing over it a grid and measuring by hand sufficient points on the graph to get within say 20% accuracy for the area.]

## Bibliography

Due to limited space available in this series, it is not possible to go into the material in great depth, so I have attempted to encapsulate what I consider to be the essentials of electrostatics and magnetism. In a separate volume to be published by Ventus Publishing ApS, “Essential Electrodynamics”, I cover in similar depth Maxwell’s equations, electromagnetic waves and radiation. Both books contain student exercises, and I give the solutions to these exercises in separate volumes also to be published by Ventus: “Essential Electromagnetism - Solutions” and “Essential Electrodynamics - Solutions”.

My aim with the present book is not to replace existing textbooks in these subjects of which there are many excellent examples, but rather to supplement the textbook specified by the professor teaching the course — I have always found the need to read about a particular topic in at least two books to improve my understanding. Nevertheless, if appropriately supplemented, this book together with “Essential Electrodynamics” could serve as a textbook for 2nd and 3rd year electromagnetism courses at Australian and British universities, or for junior/senior level electromagnetism courses at American universities/colleges.

A few of the books that I found helpful over many years of teaching electromagnetism are listed below.

- Barger, VD & Olsson MG 1987 *Classical electricity and magnetism: a contemporary perspective*, Allyn and Bacon, Boston
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- Lorrain, P, Corson, DR & Lorrain, F 1988, *Electromagnetic fields and waves: including electric circuits*, 3rd Edn. Freeman, New York
- Nayfeh, MH & Brussel, MK 1985, *Electricity and Magnetism*, Wiley, New York
- Purcell EM 1965, *Electricity and Magnetism*, McGraw-Hill, New York

A useful free resource of physical and chemical data is available at

- Kaye, GWC & Laby, TH 1995, *Tables of Physical & Chemical Constants*, 16th Edn. Kaye & Laby Online [www.kayelaby.npl.co.uk](http://www.kayelaby.npl.co.uk)

The following websites provide online integration, differentiation, etc.

- *Wolfram Mathematica online integrator*, <http://integrals.wolfram.com/index.jsp>
- *WolframAlpha computational knowledge engine*, <http://www.wolframalpha.com/>



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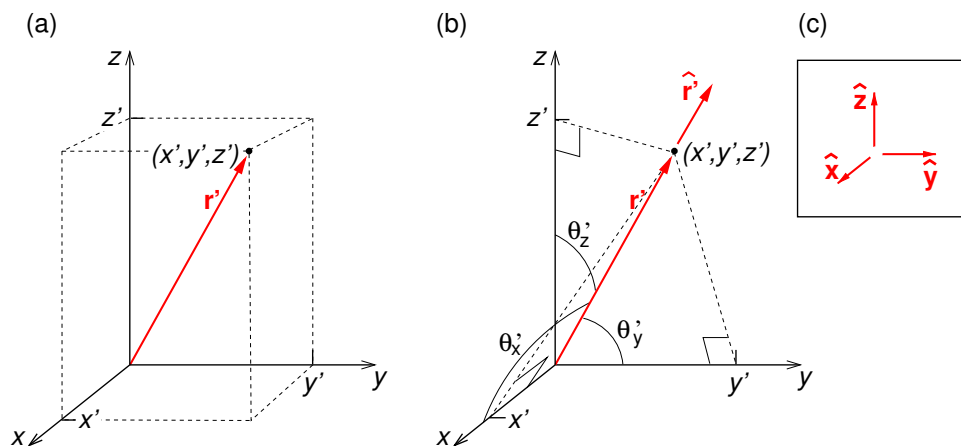
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## Appendices

### A Vectors

Vectors have both a magnitude and a direction. In print they are represented by a letter in boldface type, e.g. the magnetisation might be written  $\mathbf{M}$ , or by an arrow over the symbol when handwritten, e.g.  $\vec{M}$ . In three-dimensional space we can have vectors such as the velocity of a particle at time  $t$ , say  $\mathbf{v}(t)$ , and vector fields such as the flow velocity within a fluid  $\mathbf{v}(x, y, z, t)$  at position  $(x, y, z)$  at time  $t$ . One can also have scalars such as the charge  $q$  of a particle, and scalar fields such as charge density  $\rho(x, y, z, t)$  at position  $(x, y, z)$  at time  $t$ . What scalars and scalar fields have in common is that they have just one component (it takes only one number to define them), whereas both vectors and vector fields have a magnitude and a direction, and so it takes three numbers to define them.

Perhaps the simplest vector to describe is the position vector  $\mathbf{r}$  of some point in space defined by its Cartesian coordinates  $(x, y, z)$ , or the vector  $\mathbf{r}'$  being the position vector corresponding to the point  $(x', y', z')$ . The position vector  $\mathbf{r}'$  is represented by an arrow extending from the origin  $(0, 0, 0)$  to  $(x', y', z')$  as in Fig. A.1(a). Its magnitude is just the length (measured in m) of the arrow which, using Pythagoras' theorem, is  $r' = \sqrt{x'^2 + y'^2 + z'^2}$ . The vector's direction is represented by three *direction cosines*,  $\cos \theta'_x$ ,  $\cos \theta'_y$  and  $\cos \theta'_z$ , which are the cosines of the angles the vector makes with the three Cartesian axes as shown in Fig. A.1(b).



**Figure A.1:** (a) Position vector  $\mathbf{r}'$  of point  $(x', y', z')$ . (b) Position vector  $\mathbf{r}'$  and the angles  $\theta'_x$ ,  $\theta'_y$  and  $\theta'_z$ ; note that  $x' = r' \cos \theta'_x$ ,  $y' = r' \cos \theta'_y$  and  $z' = r' \cos \theta'_z$ . (c) Unit vectors in the  $x$ ,  $y$  and  $z$  directions.

We define a *unit vector* as being a vector having a dimensionless magnitude of 1. We have

just seen one example — the direction cosines of  $\mathbf{r}'$  are the Cartesian components of the unit vector in the direction of  $\mathbf{r}'$ , i.e.  $\hat{\mathbf{r}}' = (\cos \theta'_x, \cos \theta'_y, \cos \theta'_z)$ . Note that the “hat” above vector  $\mathbf{r}'$  indicates that it is a unit vector, i.e. in this case

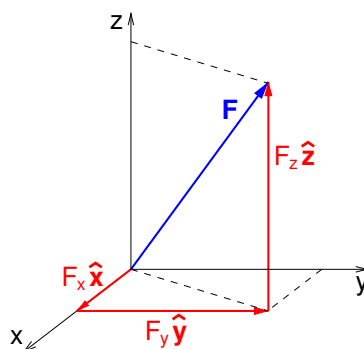
$$\hat{\mathbf{r}}' = \frac{\mathbf{r}'}{|\mathbf{r}'|} = \frac{(x', y', z')}{\sqrt{x'^2 + y'^2 + z'^2}} = (\cos \theta'_x, \cos \theta'_y, \cos \theta'_z). \quad (\text{A.1})$$

A unit vector  $\hat{\mathbf{r}}'$  pointing in the direction of  $\mathbf{r}'$  has been added to Fig. A.1(b). Unit vectors in the  $x$ ,  $y$  and  $z$  directions are written  $\hat{\mathbf{x}}$ ,  $\hat{\mathbf{y}}$  and  $\hat{\mathbf{z}}$  (in some books they are written  $\mathbf{i}$ ,  $\mathbf{j}$  and  $\mathbf{k}$ ). These three unit vectors form an orthogonal set of basis vectors (Fig. A.1c).

The force  $\mathbf{F}$  on a particle has a magnitude  $F$  (measured in N) and acts in the direction defined by its direction cosines. Multiplying the magnitude of the force by the three direction cosines gives the three Cartesian *components* of the force,  $F_x$ ,  $F_y$  and  $F_z$ , each of which is a scalar, and these three numbers (together with the vector’s dimensions) completely specify the vector. Multiplying the three components of  $\mathbf{F}$  by the corresponding unit vectors we can reconstruct the original vector by *vector addition*,

$$\mathbf{F} = F_x \hat{\mathbf{x}} + F_y \hat{\mathbf{y}} + F_z \hat{\mathbf{z}}, \quad (\text{A.2})$$

where the rule for vector addition may be stated as: “Put the 2nd vector so that its tail is at the head of the 1st arrow, then the 3rd arrow so its tail is at the head of the 2nd, and the vector sum of the three vectors is the vector extending from the tail of the 1st vector to the head of the 3rd vector” (see Fig. A.2). Notice that we obtained each of the three vectors that we added by multiplying a vector, e.g.  $\hat{\mathbf{x}}$ , by a scalar,  $F_x$ , to get a new vector  $F_x \hat{\mathbf{x}}$  parallel to the  $\hat{\mathbf{x}}$  and having magnitude equal to the magnitude of  $\hat{\mathbf{x}}$  (1 because it is a unit vector) multiplied by the value of the scalar ( $F_x$ ).



**Figure A.2:** The components of  $\mathbf{F}$ .

## A.1 Dot and cross products

Multiplying vectors by vectors is a little more complicated as there are different types of multiplication to consider. We have just now implicitly used the scalar product or “dot product” of two vectors, written

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta \quad (\text{A.3})$$

where  $\theta$  is the angle between the directions of the two vectors. Taking the dot product of vector  $\mathbf{F}$  with a unit vector gives the component of  $\mathbf{F}$  in the direction of the unit vector, e.g.

$$F_x = \mathbf{F} \cdot \hat{\mathbf{x}} = F \cos \theta_x, \quad F_y = \mathbf{F} \cdot \hat{\mathbf{y}} = F \cos \theta_y, \quad F_z = \mathbf{F} \cdot \hat{\mathbf{z}} = F \cos \theta_z. \quad (\text{A.4})$$

The vector product or “cross product” of two vectors, written  $\mathbf{A} \times \mathbf{B}$ , is a vector whose magnitude is

$$|\mathbf{A} \times \mathbf{B}| = AB \sin \theta \quad (\text{A.5})$$

where  $\theta$  is the angle between the directions of the two vectors, and its direction is given by the right hand rule (Fig. A.3). Hence, the Cartesian unit vectors obey the following

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \quad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \quad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}, \quad \hat{\mathbf{x}} \times \hat{\mathbf{z}} = -\hat{\mathbf{y}}, \quad \hat{\mathbf{y}} \times \hat{\mathbf{x}} = -\hat{\mathbf{z}}, \quad \hat{\mathbf{z}} \times \hat{\mathbf{y}} = -\hat{\mathbf{x}}, \quad (\text{A.6})$$

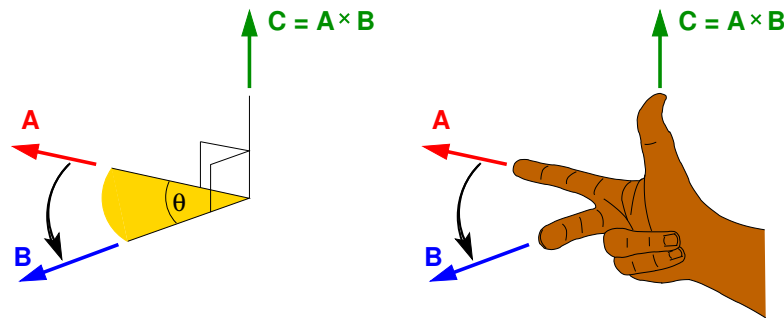
and  $\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0$ . We can use these relations to obtain a formula for the cross product of two vectors in terms of their components

$$\mathbf{A} \times \mathbf{B} = (A_x \hat{\mathbf{x}} + A_y \hat{\mathbf{y}} + A_z \hat{\mathbf{z}}) \times (B_x \hat{\mathbf{x}} + B_y \hat{\mathbf{y}} + B_z \hat{\mathbf{z}}), \quad (\text{A.7})$$

$$= 0 + A_x B_y \hat{\mathbf{z}} - A_x B_z \hat{\mathbf{y}} + -A_y B_x \hat{\mathbf{z}} + 0 + A_y B_z \hat{\mathbf{x}} + A_z B_x \hat{\mathbf{y}} - A_z B_y \hat{\mathbf{x}} + 0, \quad (\text{A.8})$$

$$= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}. \quad (\text{A.9})$$

Two separation vectors which start at the same point, such as  $\mathbf{A}$  and  $\mathbf{B}$  in Fig. A.4(a), generate



**Figure A.3:** The cross product and the right hand rule – point your index finger in the direction of the first vector (**A** in this case) and swing your 2nd finger to point in the direction of the 2nd vector (**B** in this case) and then your thumb is pointing in the direction of their cross product ( $\mathbf{A} \times \mathbf{B}$ ).

a parallelogram which has an area equal to  $|\mathbf{A} \times \mathbf{B}|$ . Three separation vectors which start at the same point, such as **A**, **B** and **C** in Fig. A.4(b), generate a parallelepiped. The volume of this parallelepiped (also called the *scalar triple product*) is

$$\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}). \quad (\text{A.10})$$

The area of the parallelogram generated by two separation vectors, and the volume of the

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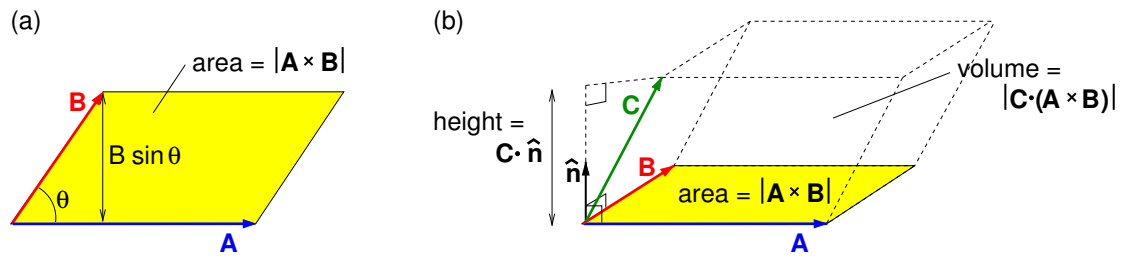
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parallelepiped generated by three separation vectors are useful for constructing the infinitesimal surface and volume elements in curvilinear coordinate systems, i.e. Jacobian determinants.



**Figure A.4:** (a) Cross product of two vectors generates a parallelogram of area  $|\mathbf{A} \times \mathbf{B}|$ .  
 (b) Scalar triple product of three vectors generates a parallelepiped of volume  $|\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})|$ .  
 ( $\hat{\mathbf{n}}$  is a unit vector normal to the parallelogram and is in the direction of  $\mathbf{A} \times \mathbf{B}$ .)

## A.2 Vector product identities

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}), \quad (\text{A.11})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}). \quad (\text{A.12})$$

## B Vector differential calculus

### B.1 The Gradient

In one dimension the differential operator  $d/dx$  operating on a function  $f(x)$  gives  $df/dx$ , i.e. the gradient at  $x$  of a plot of  $f$  vs.  $x$ . In three dimensions the gradient operator  $\nabla$  operating on a scalar field  $f(x, y, z)$  gives a vector field whose magnitude is the steepest gradient, i.e. highest rate of increase of  $f$  with distance, and is pointing in the direction of the steepest gradient. The gradient operator is

$$\nabla = \left( \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right). \quad (\text{B.1})$$

A useful example is the scalar field  $r^n = (x^2 + y^2 + z^2)^{n/2}$ ,

$$\nabla r^n = \hat{\mathbf{x}} \frac{\partial}{\partial x} (x^2 + y^2 + z^2)^{n/2} + \hat{\mathbf{y}} \frac{\partial}{\partial y} (x^2 + y^2 + z^2)^{n/2} + \hat{\mathbf{z}} \frac{\partial}{\partial z} (x^2 + y^2 + z^2)^{n/2}, \quad (\text{B.2})$$

$$\begin{aligned} &= \hat{\mathbf{x}} \frac{n}{2} (x^2 + y^2 + z^2)^{(n/2-1)} 2x + \hat{\mathbf{y}} \frac{n}{2} (x^2 + y^2 + z^2)^{(n/2-1)} 2y \\ &\quad + \hat{\mathbf{z}} \frac{n}{2} (x^2 + y^2 + z^2)^{(n/2-1)} 2z, \end{aligned} \quad (\text{B.3})$$

$$= n (\hat{\mathbf{x}}x + \hat{\mathbf{y}}y + \hat{\mathbf{z}}z) r^{n-2}, \quad (\text{B.4})$$

$$= n \mathbf{r} r^{n-2}, \quad (\text{B.5})$$

$$= n r^{n-1} \hat{\mathbf{r}}. \quad (\text{B.6})$$

Similarly, for the  $n$ th power of the magnitude of the separation vector

$$R^n = |\mathbf{r} - \mathbf{r}'| = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{n/2},$$

$$\nabla R^n = n R^{n-1} \hat{\mathbf{R}}. \quad (\text{B.7})$$

We can also differentiate  $R^n$  with respect to primed coordinates, i.e. using the operator

$$\nabla' = \left( \hat{\mathbf{x}} \frac{\partial}{\partial x'} + \hat{\mathbf{y}} \frac{\partial}{\partial y'} + \hat{\mathbf{z}} \frac{\partial}{\partial z'} \right) \quad (\text{B.8})$$

with the result

$$\nabla' R^n = -n R^{n-1} \hat{\mathbf{R}}. \quad (\text{B.9})$$

## B.2 The Divergence

The divergence operator is

$$\nabla \cdot \equiv \left( \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right). \quad (\text{B.10})$$

and it operates on a vector field, producing a scalar field which measures the density of the whatever is the source of the vector field. The divergence of the vector field  $\mathbf{A}(x, y, z)$  is

$$\nabla \cdot \mathbf{A} = \left( \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}. \quad (\text{B.11})$$

As an example, the divergence of  $\mathbf{A}(x, y, z) = 2x\hat{x} + xyz^2\hat{y} + (x^7 + y + z)\hat{z}$  is

$$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(x^7 + y + z) = xz^2 + 3. \quad (\text{B.12})$$

### B.3 The Curl

The curl operator is

$$\nabla \times \equiv \left( \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) \times \quad (\text{B.13})$$

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and it operates on a vector field and produces a vector field which measures the rotation (curling) present in the vector field at the point where it is evaluated. In Cartesian coordinates its components give the amount of rotation around axes parallel to the  $x$ ,  $y$  and  $z$  directions,

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}, \quad (\text{B.14})$$

$$\therefore \nabla \times \mathbf{A} = \hat{x} \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{y} \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{z} \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right). \quad (\text{B.15})$$

## C Vector integral calculus

### C.1 Line integrals

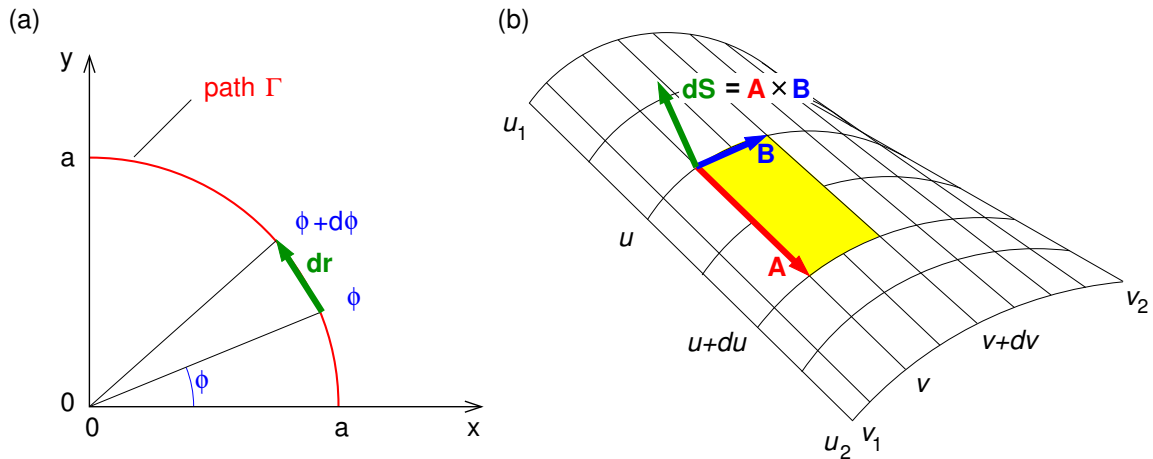
A line integral (or path integral) is an integral of some quantity along some specified path in space. A path is one-dimensional and so locations along the path  $\mathbf{r}(\lambda)$  may be uniquely specified by a single parameter, in this case  $\lambda$ . As one moves along the path increasing  $\lambda$  by  $d\lambda$  the position vector changes from  $\mathbf{r}$  to  $(\mathbf{r} + d\mathbf{r})$  where  $d\mathbf{r} = (d\mathbf{r}/d\lambda)d\lambda$ . For example, the quarter circle shown in Fig. C.1(a) may be parameterised in terms of the angle  $\phi$ ,

$$\mathbf{r}(\phi) = a \cos \phi \hat{\mathbf{x}} + a \sin \phi \hat{\mathbf{y}}. \quad (\text{C.1})$$

As one moves along the path increasing  $\phi$  by  $d\phi$  the position vector changes from  $\mathbf{r}$  to  $(\mathbf{r} + d\mathbf{r})$  where

$$d\mathbf{r} = \frac{d\mathbf{r}}{d\phi} d\phi = \left( \hat{\mathbf{x}} \frac{dx}{d\phi} + \hat{\mathbf{y}} \frac{dy}{d\phi} + \hat{\mathbf{z}} \frac{dz}{d\phi} \right) d\phi = (-a \sin \phi \hat{\mathbf{x}} + a \cos \phi \hat{\mathbf{y}} + 0) d\phi. \quad (\text{C.2})$$

The line integral may be of several different types. For example  $\int_{\Gamma, \mathbf{r}_1}^{\mathbf{r}_2} |d\mathbf{r}|$  would simply give the distance along path  $\Gamma$  (from  $\mathbf{r}_1$  to  $\mathbf{r}_2$ ), and  $\int_{\Gamma, \mathbf{r}_1}^{\mathbf{r}_2} \mathbf{A} \cdot d\mathbf{r}$  would give the integral of the tangential component of  $\mathbf{A}$  along the path  $\Gamma$ . For example, the line integral of  $\mathbf{A} = bx \hat{\mathbf{x}}$  around the



**Figure C.1:** (a) A line element  $d\mathbf{r}$  in a quarter circle in the 1st quadrant of the  $xy$  plane. (b) Surface element  $d\mathbf{S}$  in a surface defined by  $\mathbf{r}(u, v)$  for  $u_1 < u < u_2$  and  $v_1 < v < v_2$ .

quarter circle is

$$\int_{\Gamma} \mathbf{A} \cdot d\mathbf{r} = \int_0^{\pi/2} b(a \cos \phi) \hat{\mathbf{x}} \cdot (-a \sin \phi \hat{\mathbf{x}} + a \cos \phi \hat{\mathbf{y}}) d\phi, \quad (\text{C.3})$$

$$= -ba^2 \int_0^{\pi/2} \sin \phi \cos \phi d\phi = -ba^2 \left[ -\frac{1}{2} \cos^2 \phi \right]_0^{\pi/2} = -\frac{ba^2}{2}. \quad (\text{C.4})$$

A line integral around a closed loop is represented by  $\oint_{\Gamma} \mathbf{A} \cdot d\mathbf{r}$ .

## C.2 Surface Integrals

The most common type of surface integral encountered in electromagnetism is the flux integral which is of the form  $\int_S \mathbf{F} \cdot d\mathbf{S}$  and gives the flux of the vector field  $\mathbf{F}$  through the surface. For example if  $\mathbf{F}$  represented the flow velocity ( $\text{m s}^{-1}$ ) of an incompressible fluid then the flux integral would give the rate at which the fluid pass through the surface ( $\text{m}^3 \text{s}^{-1}$ ).

We need two parameters, say  $u$  and  $v$  to parameterise a surface  $S$ ,  $\mathbf{r} = \mathbf{r}(u, v)$ . Then the surface element can be written  $d\mathbf{S} = \mathbf{A} \times \mathbf{B}$ , as in Fig. C.1(b), where

$$\mathbf{A} = \frac{\partial \mathbf{r}}{\partial u} du, \quad \mathbf{B} = \frac{\partial \mathbf{r}}{\partial v} dv, \quad (\text{C.5})$$

Hence, for  $S$  corresponding to  $u_1 < u < u_2$  and  $v_1 < v < v_2$

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_{u_1}^{u_2} \int_{v_1}^{v_2} \mathbf{F}(u, v) \cdot \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} dv du. \quad (\text{C.6})$$

There are two directions perpendicular to any surface, so the direction must be specified – by definition it is outward from the volume for integrals over closed surfaces, which are written as  $\oint_S \mathbf{F} \cdot d\mathbf{S}$ . One must always make sure that  $d\mathbf{S}$  points in the right direction for the problem (i.e. into or out of the surface).

As an example, we shall consider a simple flux integral over the surface of the cone shown in Fig. C.2. For points in the surface

$$\mathbf{r} = r \sin \theta_c \cos \phi \hat{\mathbf{x}} + r \sin \theta_c \sin \phi \hat{\mathbf{y}} + r \cos \theta_c \hat{\mathbf{z}}. \quad (\text{C.7})$$

and so it makes sense to use as parameters  $r$  ( $0 < r \leq R$ ) and  $\phi$  ( $0 < \phi \leq 2\pi$ ).

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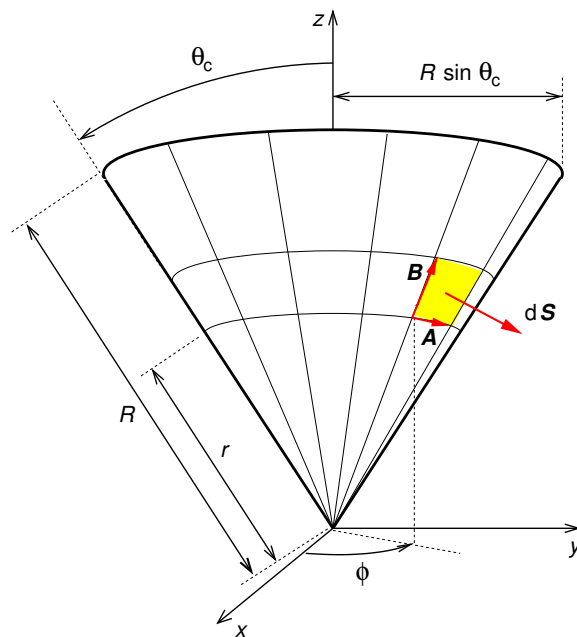
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**Figure C.2:** Surface element  $d\mathbf{S}$  in the surface of a cone.

Then

$$\mathbf{A} = \frac{\partial \mathbf{r}}{\partial \phi} d\phi = -r \sin \theta_c \sin \phi d\phi \hat{\mathbf{x}} + r \sin \theta_c \cos \phi d\phi \hat{\mathbf{y}}, \quad (\text{C.8})$$

$$\mathbf{B} = \frac{\partial \mathbf{r}}{\partial r} dr = \sin \theta_c \cos \phi dr \hat{\mathbf{x}} + \sin \theta_c \sin \phi dr \hat{\mathbf{y}} + \cos \theta_c dr \hat{\mathbf{z}}, \quad (\text{C.9})$$

$$d\mathbf{S} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -r \sin \theta_c \sin \phi d\phi & r \sin \theta_c \cos \phi d\phi & 0 \\ \sin \theta_c \cos \phi dr & \sin \theta_c \sin \phi dr & \cos \theta_c dr \end{vmatrix}. \quad (\text{C.10})$$

$$\therefore d\mathbf{S} = (\sin \theta_c \cos \theta \cos \phi \hat{\mathbf{x}} + \sin \theta_c \cos \theta \sin \phi \hat{\mathbf{y}} - \sin^2 \theta_c \hat{\mathbf{z}}) r dr d\phi. \quad (\text{C.11})$$

We now have the surface element, and for our simple example of a flux integral we will integrate

$\mathbf{F}(\mathbf{r}) = F \hat{\mathbf{z}}$  over this surface,

$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{S} &= \int_0^{2\pi} \int_0^R (F \hat{\mathbf{z}}) \cdot (\sin \theta_c \cos \theta \cos \phi \hat{\mathbf{x}} + \sin \theta_c \cos \theta \sin \phi \hat{\mathbf{y}} - \sin^2 \theta_c \hat{\mathbf{z}}) r dr d\phi, \\ &= -F \sin^2 \theta_c \int_0^{2\pi} \int_0^R r dr d\phi, \end{aligned} \quad (\text{C.12})$$

$$= -F \pi R^2 \sin^2 \theta_c. \quad (\text{C.13})$$

This is obviously correct since  $\mathbf{F} = F \hat{\mathbf{z}}$  is constant,  $d\mathbf{S} \cdot \hat{\mathbf{z}}$  is everywhere negative and  $\pi(R \sin \theta_c)^2$  is the projected area of the cone perpendicular to  $\hat{\mathbf{z}}$ . In this case we didn't need to parameterise the surface or do the integral; we could have just inspected the problem and written down the answer!

### C.3 Volume Integrals

For volume integrals we need three parameters,  $u, v$ , and  $w$ . The volume element is  $dV = |\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})|$  as in Fig. A.4(b) with

$$\mathbf{C} = \frac{\partial \mathbf{r}}{\partial u} du, \quad \mathbf{A} = \frac{\partial \mathbf{r}}{\partial v} dv, \quad \mathbf{B} = \frac{\partial \mathbf{r}}{\partial w} dw. \quad (\text{C.14})$$

Then

$$\int_V F d^3r = \int \int \int F |J| du dv dw \quad (\text{C.15})$$

where  $J$  is the Jacobian determinant, written

$$J \equiv \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{\partial \mathbf{r}}{\partial u} \cdot \left( \frac{\partial \mathbf{r}}{\partial v} \times \frac{\partial \mathbf{r}}{\partial w} \right). \quad (\text{C.16})$$

When symmetry permits one can often use cylindrical or spherical polar coordinates for which the Jacobian determinants are  $J = \rho$  (cylindrical coordinates  $\rho, \phi, z$ ), and  $J = r^2 \sin \theta$  (spherical coordinates  $r, \theta, \phi$ ).



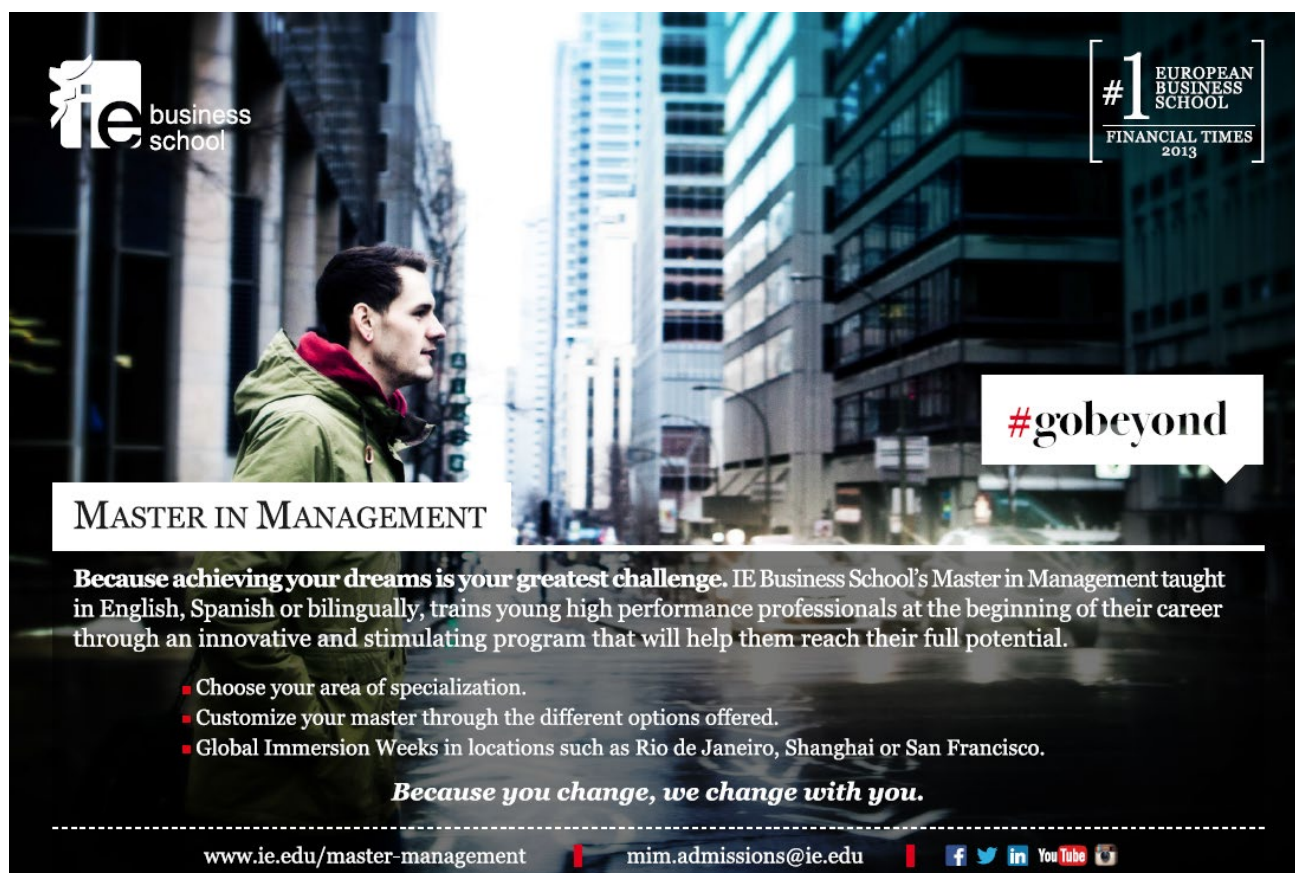
## C.4 Gradient theorem

The gradient theorem states

$$\int_{\mathbf{r}_a}^{\mathbf{r}_b} \nabla \phi \cdot d\mathbf{r} = \phi(\mathbf{r}_b) - \phi(\mathbf{r}_a) \quad (\text{C.17})$$

independent of path, as is easily proved from the definition of the gradient,  $\nabla \phi \cdot d\mathbf{r} = d\phi$ . This is an exact differential, and so

$$\int_{\mathbf{r}_a}^{\mathbf{r}_b} \nabla \phi \cdot d\mathbf{r} = \int_{\mathbf{r}_a}^{\mathbf{r}_b} d\phi = [\phi]_{\mathbf{r}_a}^{\mathbf{r}_b} = \phi(\mathbf{r}_b) - \phi(\mathbf{r}_a). \quad (\text{C.18})$$



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## C.5 Gauss' theorem

Gauss' Theorem (Divergence Theorem) states

$$\int_V \nabla \cdot \mathbf{F} d^3r = \oint_S \mathbf{F} \cdot d\mathbf{S} \quad (\text{C.19})$$

where  $d\mathbf{S}$  is the surface normal which is directed outwards from the volume, and surface  $S$  bounds volume  $V$ . So we could in fact define the divergence of  $\mathbf{F}$  as

$$\nabla \cdot \mathbf{F} = \lim_{V \rightarrow 0} \left( \frac{1}{V} \oint_S \mathbf{F} \cdot d\mathbf{S} \right), \quad (\text{C.20})$$

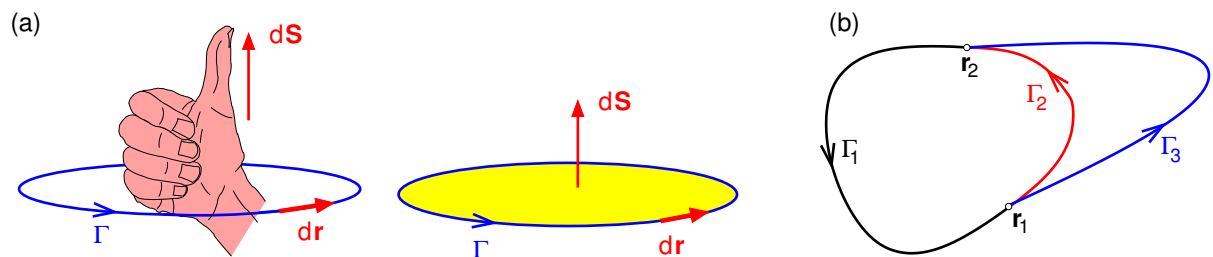
and so the divergence of a vector field  $\mathbf{F}(\mathbf{r})$  can be thought of as the net outward flux of  $\mathbf{F}$  per unit volume.

## C.6 Stokes' theorem

Stokes' theorem (curl theorem) states that

$$\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_\Gamma \mathbf{F} \cdot d\mathbf{r} \quad (\text{C.21})$$

where loop  $\Gamma$  bounds surface  $S$ , and  $d\mathbf{r}$  is directed depending on the orientation of  $d\mathbf{S}$  according to the right hand rule for use with Stokes' theorem as in Fig. C.3(a) — fingers curl around path  $\Gamma$  and thumb points in direction  $d\mathbf{S}$ .



**Figure C.3:** (a) Right hand rule for directions of line integration and surface element when using Stokes' theorem. (b) Three paths joining  $r_1$  to  $r_2$ .

### C.6.1 Conservative fields and exactness

If for a vector field  $\mathbf{F}(\mathbf{r})$  one finds that  $\nabla \times \mathbf{F} = 0$ , then  $\mathbf{F}$  is said to be *conservative*, and Stokes' theorem implies that

$$\oint_{\Gamma} \mathbf{F} \cdot d\mathbf{r} = 0 \quad (\text{C.22})$$

for any closed loop such as  $(\Gamma_1 + \Gamma_2)$  or  $(\Gamma_1 + \Gamma_3)$  in Fig. C.3(b). Hence, if  $\mathbf{F}$  is conservative

$$\oint_{(\Gamma_1 + \Gamma_2)} \mathbf{F} \cdot d\mathbf{r} = \oint_{(\Gamma_1 + \Gamma_3)} \mathbf{F} \cdot d\mathbf{r} = 0 \quad (\text{C.23})$$

Since  $\Gamma_1$  is common to both these closed loops

$$\int_{\mathbf{r}_1, \Gamma_2}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathbf{r}_1, \Gamma_3}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} \quad (\text{C.24})$$

and so the integral must be independent of path. This allows us to define a potential

$$\Phi(\mathbf{r}) = - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}' \quad (\text{C.25})$$

such that  $\mathbf{F}(\mathbf{r}) = -\nabla\Phi$ . Then

$$- \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}' = \int_{\mathbf{r}_0}^{\mathbf{r}} \nabla\Phi(\mathbf{r}') \cdot d\mathbf{r}' = \int_{\mathbf{r}_0}^{\mathbf{r}} d\Phi = \Phi(\mathbf{r}) - \Phi(\mathbf{r}_0), \quad (\text{C.26})$$

and we see that if  $\nabla \times \mathbf{F} = 0$  then  $\mathbf{F} \cdot d\mathbf{r}$  is exact because  $\nabla\Phi(\mathbf{r}) \cdot d\mathbf{r} = d\Phi$  is an exact differential.

Before moving on, a word of caution is necessary. The result above is true only for *simply-connected* regions, i.e. regions without holes in them. An example of a *multiply-connected* region is shown in Fig. C.4(a). Everywhere along loop  $\Gamma_1$  we have  $\nabla \times \mathbf{E} = 0$  and so we might

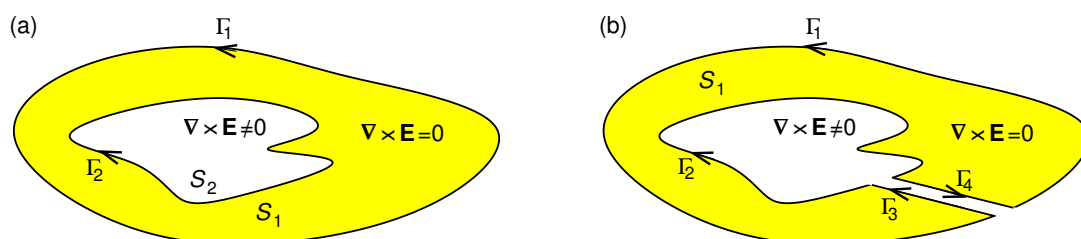
reasonably expect that  $\oint_{\Gamma_1} \mathbf{E} \cdot d\mathbf{r} = 0$ . However, it's not! Using Stokes' theorem

$$\oint_{\Gamma_1} \mathbf{E} \cdot d\mathbf{r} = \int_{S_1} \nabla \times \mathbf{E} \cdot d\mathbf{S} + \int_{S_2} \nabla \times \mathbf{E} \cdot d\mathbf{S}, \quad (\text{C.27})$$

$$= 0 + \int_{S_2} \nabla \times \mathbf{E} \cdot d\mathbf{S}, \quad (\text{C.28})$$

$$= - \oint_{\Gamma_2} \mathbf{E} \cdot d\mathbf{r} \neq 0, \quad (\text{C.29})$$

and this is because the region is multiply-connected and inside  $S_2$  we have  $\nabla \times \mathbf{E} \neq 0$ .



**Figure C.4:** (a) A multiply-connected region comprising region  $S_1$  where  $\nabla \times \mathbf{E} = 0$  surrounding a hole (region  $S_2$ ) where  $\nabla \times \mathbf{E} \neq 0$ . (b) As part (a) but with a cut to make region  $S_1$  simply connected.

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We can make region  $S_1$  simply-connected by making a cut as shown in Fig. C.4(b). Then, integrating around the boundary of the simply-connected region

$$\oint_{\Gamma_1+\Gamma_3+\Gamma_2+\Gamma_4} \mathbf{E} \cdot d\mathbf{r} = 0. \quad (\text{C.30})$$

## C.7 Dirac delta function

Point charges have infinite charge density but finite charge, and we can use the Dirac delta function,  $\delta(x - x_0)$ , to represent them. The one-dimensional version  $\delta(x)$  is defined such that

$$\int_{-\infty}^{\infty} f(x) \delta(x - a) dx = f(a). \quad (\text{C.31})$$

An important property is

$$\delta(cx) = \frac{1}{|c|} \delta(x). \quad (\text{C.32})$$

In three dimensions we have

$$\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z), \quad (\text{C.33})$$

$$\int_{\text{all space}} \delta^3(\mathbf{r}) d^3r \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z) dx dy dz = 1, \quad (\text{C.34})$$

$$\int_{\text{all space}} f(\mathbf{r}) \delta^3(\mathbf{r} - \mathbf{a}) d^3r = f(\mathbf{a}). \quad (\text{C.35})$$

In cylindrical and spherical coordinates we need to divide by the appropriate Jacobian

$$\delta^3(\mathbf{r} - \mathbf{r}_0) = \frac{1}{r^2 \sin \theta} \delta(r - r_0) \delta(\theta - \theta_0) \delta(\phi - \phi_0) \quad (\text{spherical}), \quad (\text{C.36})$$

$$\delta^3(\mathbf{r} - \mathbf{r}_0) = \frac{1}{\rho} \delta(\rho - \rho_0) \delta(\phi - \phi_0) \delta(z - z_0) \quad (\text{cylindrical}). \quad (\text{C.37})$$

Two important identities are

$$\nabla \cdot \left( \frac{\hat{\mathbf{R}}}{R^2} \right) = 4\pi\delta^3(\mathbf{R}), \quad (\text{C.38})$$

$$\nabla^2 \frac{1}{R} = -4\pi\delta^3(\mathbf{R}), \quad (\text{C.39})$$

where  $\mathbf{R} = (\mathbf{r} - \mathbf{r}')$ .

## C.8 Heaviside step function

Discontinuous changes of some quantity with position can be represented by the Heaviside step function  $H(x - x_0)$  defined by

$$H(x - x_0) = \begin{cases} 0 & (x < x_0), \\ 1/2 & (x = x_0), \\ 1 & (x > x_0). \end{cases} \quad (\text{C.40})$$

It has the property that

$$\frac{d}{dx} H(x - x_0) = \delta(x - x_0). \quad (\text{C.41})$$

An example of its use would be to describe a square voltage pulse of height  $V_0$  starting at time  $t_1$  and ending at  $t_2$  for which

$$V(t) = V_0 H(t_1) [1 - H(t_2)]. \quad (\text{C.42})$$

Another example would be to describe the electric field on the surface of a spherical conductor of radius  $a$  carrying charge  $q$ . The radial component of the electric field is zero inside and finite and radial outside,

$$E_r(r) = \frac{q}{4\pi\epsilon_0 r^2} H(r - a). \quad (\text{C.43})$$

Gauss' law tells us that the charge density is  $\varepsilon_0 \nabla \cdot \mathbf{E}$

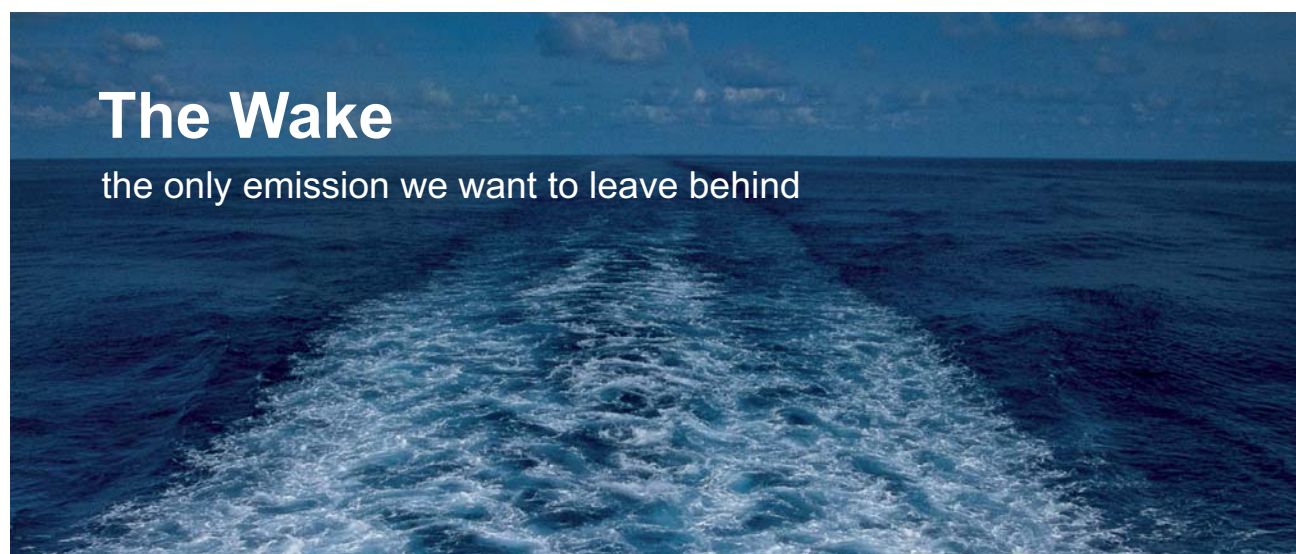
$$\rho(r) = \varepsilon_0 \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r), \quad (\text{C.44})$$

$$= \varepsilon_0 \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{q}{4\pi\varepsilon_0 r^2} H(r-a) \right), \quad (\text{C.45})$$

$$= \frac{q}{4\pi r^2} \frac{\partial}{\partial r} H(r-a), \quad (\text{C.46})$$

$$= \frac{q}{4\pi r^2} \delta(r-a), \quad (\text{C.47})$$

where  $\delta(x)$  is the Dirac delta function. We know that all the charge resides on the surface of a conductor, so that in this case we have a surface charge density  $\sigma(a) = q/4\pi a^2$  which corresponds to the volume charge density above.



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## D Basics of index notation

The components of the vector  $\mathbf{A}$  are denoted  $A_i$ , where  $i$  is its index, i.e.  $A_1 \equiv A_x$ ,  $A_2 \equiv A_y$  and  $A_3 \equiv A_z$ . Similarly, for a matrix,  $B_{ij}$  represents the component in  $i$ th row and  $j$ th column.

The *Kronecker delta* is defined

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases} \quad (\text{D.1})$$

The *Levi-Civita tensor* is defined

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } i j k = 1 2 3 \text{ or } 3 1 2 \text{ or } 2 3 1 \\ -1, & \text{if } i j k = 1 3 2 \text{ or } 2 1 3 \text{ or } 3 2 1 \\ 0, & \text{otherwise} \end{cases} \quad (\text{D.2})$$

Note that  $\varepsilon_{ijk}$  is 1 for cyclic permutations of 123, and  $-1$  for anti-cyclic permutations, e.g. 321, and 0 otherwise.

In matrix multiplication  $[A] = [B][C]$

$$A_{ij} = \sum_{k=1}^3 B_{ik} C_{kj}. \quad (\text{D.3})$$

In the *Einstein summation convention* if an index appears twice (*dummy index*) then the repeated index is summed over for all three values of that index — there is no need to write the summation sign — so that

$$A_{ij} = B_{ik} C_{kj} \quad (\text{D.4})$$

is entirely equivalent to Eq. D.3. We see that the dummy index  $k$  has disappeared while the *free indices*  $i$  and  $j$  remain.

In index notation the dot product is

$$\mathbf{A} \cdot \mathbf{B} = A_i B_i. \quad (\text{D.5})$$



Following each summation the number of indices is reduced by one (in the dot product above it is reduced from 1 to 0); this is called *contraction*. Another example of contraction is

$$\delta_{ij} A_{ik} = A_{jk} \quad (\text{D.6})$$

where this time the number of indices goes from 3 to 2.

*Symmetric tensors* obey

$$C_{ij} = C_{ji} \quad \text{or} \quad C_{ijk} = C_{kji} \quad (\text{D.7})$$

and are unchanged when any two indices are swapped. *Anti-symmetric tensors* obey

$$D_{ij} = -D_{ji} \quad \text{or} \quad D_{ijk} = -D_{kji} \quad (\text{D.8})$$

and when any two indices are swapped they change sign. The Levi-Civita Tensor,  $\varepsilon_{ijk}$ , is anti-symmetric. The Kronecker Delta,  $\delta_{ij}$ , is symmetric. When you sum over all elements of the product of two tensors one of which is symmetric and the other anti symmetric the result is zero, e.g. if  $A_{ij}$  is symmetric and  $B_{ij}$  antisymmetric, then  $A_{ij}B_{ij} = 0$ .

The gradient operator is

$$\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right),$$

so that in index notation it is  $\nabla_i = \partial/\partial r_i$ .

The divergence of a vector field  $\mathbf{D}$  is

$$\nabla \cdot \mathbf{D} = \nabla_i D_i. \quad (\text{D.9})$$

The cross product of  $\mathbf{F}$  and  $\mathbf{G}$ , and its  $i$ th component are

$$\mathbf{F} \times \mathbf{G} = \varepsilon_{ijk} \hat{\mathbf{e}}_i F_j G_k, \quad [\mathbf{F} \times \mathbf{G}]_i = \varepsilon_{ijk} F_j G_k, \quad (\text{D.10})$$

where  $\hat{\mathbf{e}}_1$ ,  $\hat{\mathbf{e}}_2$  and  $\hat{\mathbf{e}}_3$  are the unit vectors in the  $x$ ,  $y$  and  $z$  directions.

Similarly for the Curl of a vector field  $\mathbf{H}$

$$\nabla \times \mathbf{H} = \varepsilon_{ijk} \hat{\mathbf{e}}_i \nabla_j H_k, \quad [\nabla \times \mathbf{H}]_i = \varepsilon_{ijk} \nabla_j H_k. \quad (\text{D.11})$$

A useful identity for the Kroneker delta is

$$\nabla_i r_j = \frac{\partial r_i}{\partial r_j} = \delta_{ij} \quad (\text{D.12})$$

A useful Levi-Civita Identity is

$$\varepsilon_{ijk} \varepsilon_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}. \quad (\text{D.13})$$

To make the 1st index of two Levi-Civita tensors the same apply a cyclic permutation to the indices of one of them.

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## E Uniqueness of solutions of Poisson's equation

We will prove by contradiction that the solution of Poisson's equation is unique. Consider the possibility that there could be two different solutions  $V_1(\mathbf{r})$  and  $V_2(\mathbf{r})$  for the same charge density  $\rho(\mathbf{r})$  and boundary conditions. The corresponding electric fields  $\mathbf{E}_1(\mathbf{r}) = -\nabla V_1(\mathbf{r})$  and  $\mathbf{E}_2(\mathbf{r}) = -\nabla V_2(\mathbf{r})$  must satisfy Gauss law,

$$\nabla \cdot [-\nabla V_1(\mathbf{r})] = \frac{\rho(\mathbf{r})}{\varepsilon_0} \quad \text{and} \quad \nabla \cdot [-\nabla V_2(\mathbf{r})] = \frac{\rho(\mathbf{r})}{\varepsilon_0}. \quad (\text{E.1})$$

$$\therefore \nabla \cdot [\nabla(V_2 - V_1)] = 0. \quad (\text{E.2})$$

Using product rule Eq. I.8 we may write

$$\begin{aligned} \nabla \cdot [(V_2 - V_1) \nabla(V_2 - V_1)] &= \nabla(V_2 - V_1) \cdot \nabla(V_2 - V_1) + (V_2 - V_1) \nabla \cdot [\nabla(V_2 - V_1)]. \\ \therefore \nabla \cdot [(V_2 - V_1) \nabla(V_2 - V_1)] &= |\nabla(V_2 - V_1)|^2 + 0. \end{aligned} \quad (\text{E.3})$$

Integrating over volume  $\mathcal{V}$  inside the boundary surface  $S$ ,

$$\int_{\mathcal{V}} \nabla \cdot [(V_2 - V_1) \nabla(V_2 - V_1)] d^3r = \int_{\mathcal{V}} |\nabla(V_2 - V_1)|^2 d^3r, \quad (\text{E.4})$$

$$\therefore \oint_S (V_2 - V_1) \nabla(V_2 - V_1) \cdot \hat{\mathbf{n}} dS = \int_{\mathcal{V}} |\nabla(V_2 - V_1)|^2 d^3r \quad (\text{E.5})$$

where we have used Gauss' theorem on the left hand side.

If  $V$  is specified on the boundary surface then on that surface  $V_2(\mathbf{r}) = V_1(\mathbf{r})$ , or if  $\mathbf{E} \cdot \hat{\mathbf{n}}$  is specified on the boundary  $(\nabla V_2) \cdot \hat{\mathbf{n}} = (\nabla V_1) \cdot \hat{\mathbf{n}}$ . In either case the surface integral is zero, and we are left with

$$\int_{\mathcal{V}} |\nabla(V_2 - V_1)|^2 d^3r = 0. \quad (\text{E.6})$$

Since the integrand is everywhere greater than or equal to zero, this equation can only be satisfied if  $\nabla(V_2 - V_1) = 0$ , in other words if  $\mathbf{E}_2(\mathbf{r}) = \mathbf{E}_1(\mathbf{r})$ . So, we have just have proved that the solution for the electric field is unique, and consequently that  $V_2(\mathbf{r}) = V_1(\mathbf{r}) + V_{\text{const}}$  where  $V_{\text{const}}$  is an arbitrary constant potential. If the potential is specified anywhere on the boundary then  $V_2(\mathbf{r}) = V_1(\mathbf{r})$  everywhere inside the boundary.

## F Field line tracing as an initial value problem

For solving an ordinary differential equation,  $dy/dx = f(x)$ , given initial conditions  $y(x_0) = y_0$ , one can use Euler's method with “steps” of length  $\Delta$  in  $x$ ,

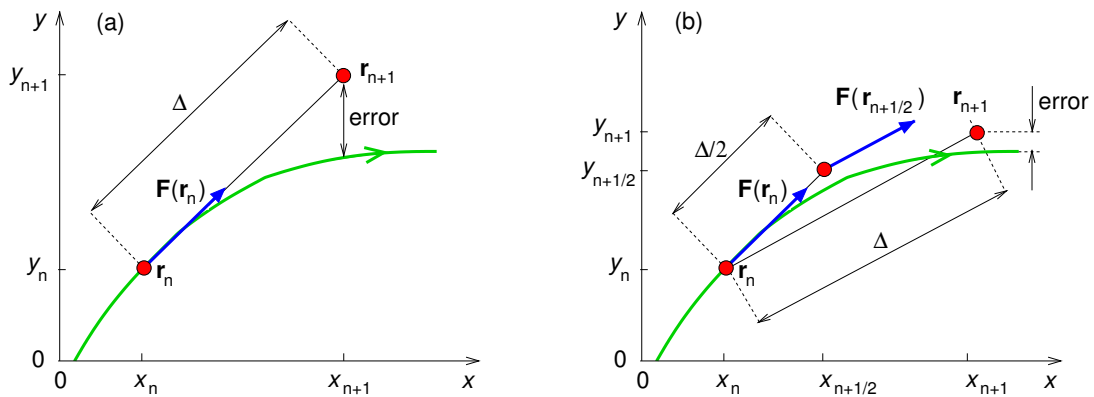
$$x_{n+1} = x_n + \Delta; \quad y_{n+1} = y_n + f(x_n)\Delta. \quad (\text{F.1})$$

Similarly for tracing field lines of vector field  $\mathbf{F}(\mathbf{r})$  one could use a scheme motivated by Euler's method, as illustrated in Fig. F.1(a)

$$\mathbf{r}_{n+1} = \mathbf{r}_n + \hat{\mathbf{F}}(\mathbf{r})\Delta. \quad (\text{F.2})$$

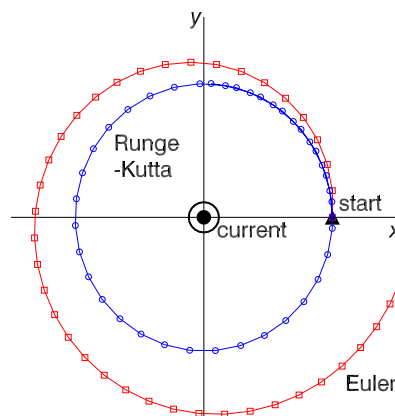
A scheme motivated by the 2nd order Runge-Kutta Method for solving initial value problems gives greatly improved accuracy, as illustrated in Fig. F.1(b)

$$\mathbf{r}_{n+1/2} = \mathbf{r}_n + \hat{\mathbf{F}}(\mathbf{r}_n)\frac{\Delta}{2}; \quad \mathbf{r}_{n+1} = \mathbf{r}_n + \hat{\mathbf{F}}(\mathbf{r}_{n+1/2})\Delta. \quad (\text{F.3})$$



**Figure F.1:** Comparison of (a) Euler's method and (b) the 2nd order Runge-Kutta method of solving initial value problems as applied to field-line tracing.


This straightforward technique has been used for producing all the images in this book which show electric or magnetic field lines. The best example to show the accuracy of this method is a magnetic field line of an infinite straight current, whose field is given by  $\mathbf{B} = (\mu_0 I / 2\pi\rho)\hat{\phi}$ . The field lines should therefore be circles in the  $x$ - $y$  plane, and the two methods discussed above are compared for the same step size in Fig. F.2.






**Figure F.2:** Comparison of (a) Euler's method and (b) the 2nd order Runge-Kutta method adapted for tracing the magnetic field of an infinite straight current along the  $z$  axis.

## G Code for 2D finite-difference method for Laplace's equation

The IDL code below was used to solve the 2D problem of the potential inside a rectangular pipe of square cross section (1 m by 1 m) where the potential is specified on each of the four



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```

n_iter=10000                                ;;; number of iterations

nx_cells=100                                ;;; number of cells in x
ny_cells=100                                ;;; number of cells in y
nx=nx_cells+2                               ;;; need extra grid points at cell
ny=ny_cells+2                               ;;; boundary each side for BC

x=(findgen(nx)-1)/(nx-3)                    ;;; x and y values
y=(findgen(ny)-1)/(ny-3)

Vx0=0.                                       ;;; potentials on the 4 boundary surfaces
Vx1=0.
Vy0=0.
Vy1=1.
Vstart=(Vx0+Vx1+Vy0+Vy1)/4.                ;;; Average of V over boundaries.
Potential=fltarr(nx,ny)+Vstart               ;;; Declare and initialise array for V.

BoundaryMask=fltarr(nx,ny)+1.               ;;; Set BC mask to 0 at boundary, 1 elsewhere.
BoundaryMask(0:1,*)=0.                      ;;; After each iteration the potential is
BoundaryMask(nx-2:nx-1,*)=0.                ;;; multiplied by the mask before adding
BoundaryMask(*,0:1)=0.                      ;;; the BC potential.
BoundaryMask(*,ny-2:ny-1)=0.

BoundaryPotential=fltarr(nx,ny)              ;;; Set to BC V at boundary, 0 elsewhere
BoundaryPotential(0:1,*)=Vx0
BoundaryPotential(nx-2:nx-1,*)=Vx1
BoundaryPotential(*,0:1)=Vy0
BoundaryPotential(2:nx-3,ny-2)=Vy1
BoundaryPotential(1:nx-2,ny-1)=Vy1
BoundaryPotential(1,ny-2)=0.5*(Vx0+Vy1)
BoundaryPotential(nx-2,ny-2)=0.5*(Vx1+Vy1)

FOR i_iter=1,n_iter DO BEGIN                 ;;; Iterate.
                                           ;;; Average over 4 nearby points
  Potential=(SHIFT(Potential,1,0)+SHIFT(Potential,-1,0) $
    +SHIFT(Potential,0,1)+SHIFT(Potential,0,-1))/4.
  Potential=Potential*BoundaryMask           ;;; Reset BC
  Potential=Potential+BoundaryPotential
ENDFOR

CONTOUR, Potential, X, Y, LEVELS=0.1*findgen(11),/fill

```

sides:  $V(0, y, z) = V(1, y, z) = V(x, 0, z) = 0$  V, and  $V(x, 1, z) = 1$  V. We choose a grid spacing  $L = 0.01$  m, i.e.  $100 \times 100$  cells ( $101 \times 101$  grid points). After each iteration, the potentials at the outer grid points are reset to the potentials on the boundary surfaces. At the corners where  $V$  changes discontinuously from  $V = 0$  to  $V = 1$  V we set the potential to be the average value, i.e.  $V(0, 1, z) = V(1, 1, z) = 0.5$  V. The result is contoured in Fig. 2.10. The code uses array manipulation methods available in modern programming languages rather than using do loops when computing the average potential around each grid point, and in resetting the boundary potential.

The IDL code contains numerous comments, but I will explain some of the procedures called which are specific to IDL.

FINDGEN( $n$ ) declares a floating-point array with  $n$  elements which have the values: 0,1,2,... $n-1$ .

FLTARR( $n, m$ ) declares a 2D a floating-point array with  $n \times m$  elements.

NAME( $i:j, *$ ) means the array made up of the elements of the 2D array NAME having the 1st index in the range  $i$  to  $j$  and all values of the 2nd index.

SHIFT(NAME,  $k, l$ ) means the array constructed from 2D array NAME by shifting the 1st index by  $k$  and the 2nd by  $l$ .

CONTOUR, together with some other formatting statements (not shown), contours the result.

## H Dipole term of multipole expansion of vector potential

The dipole term can be written

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi r^3} \sum_{i=1}^n I_i \oint_{\Gamma_i} d\mathbf{r}' \mathbf{r} \cdot \mathbf{r}'. \quad (\text{H.1})$$

Using the method of integration by parts, the integrals in the dipole term may be written as

$$\oint_{\Gamma_i} (\mathbf{r} \cdot \mathbf{r}') d\mathbf{r}' = \oint_{\Gamma_i} d[(\mathbf{r} \cdot \mathbf{r}') \mathbf{r}'] - \oint_{\Gamma_i} (\mathbf{r} \cdot d\mathbf{r}') \mathbf{r}', \quad (\text{H.2})$$

but the 1st integral on the right is zero because  $d[(\mathbf{r} \cdot \mathbf{r}')\mathbf{r}']$  is an exact differential. Hence,

$$\oint_{\Gamma_i} (\mathbf{r} \cdot \mathbf{r}') d\mathbf{r}' = - \oint_{\Gamma_i} (\mathbf{r} \cdot d\mathbf{r}') \mathbf{r}'. \quad (\text{H.3})$$

$$\therefore \oint_{\Gamma_i} (\mathbf{r} \cdot \mathbf{r}') d\mathbf{r}' + \oint_{\Gamma_i} (\mathbf{r} \cdot \mathbf{r}') d\mathbf{r}' = \oint_{\Gamma_i} (\mathbf{r} \cdot \mathbf{r}') d\mathbf{r}' - \oint_{\Gamma_i} (\mathbf{r} \cdot d\mathbf{r}') \mathbf{r}', \quad (\text{H.4})$$

$$\therefore \oint_{\Gamma_i} (\mathbf{r} \cdot \mathbf{r}') d\mathbf{r}' = \frac{1}{2} \oint_{\Gamma_i} [(\mathbf{r} \cdot \mathbf{r}') d\mathbf{r}' - (\mathbf{r} \cdot d\mathbf{r}') \mathbf{r}']. \quad (\text{H.5})$$

Now from the vector triple product rule we have  $(\mathbf{r}' \times d\mathbf{r}') \times \mathbf{r} = (\mathbf{r} \cdot \mathbf{r}') d\mathbf{r}' - (\mathbf{r} \cdot d\mathbf{r}') \mathbf{r}'$  (Appendix A.2), and so

$$\oint_{\Gamma_i} (\mathbf{r} \cdot \mathbf{r}') d\mathbf{r}' = \frac{1}{2} \oint_{\Gamma_i} (\mathbf{r}' \times d\mathbf{r}') \times \mathbf{r}. \quad (\text{H.6})$$

$$\therefore \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{r}}{r^3}, \quad (\text{H.7})$$

where the magnetic dipole moments for  $n$  current loops and a volume current distribution are

$$\mathbf{m} = \frac{1}{2} \sum_{i=1}^n I_i \oint_{\Gamma_i} \mathbf{r}' \times d\mathbf{r}', \quad \mathbf{m} = \frac{1}{2} \int \mathbf{r}' \times \mathbf{J}(\mathbf{r}') d^3r'. \quad (\text{H.8})$$

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# I Summary of vector calculus identities

## I.1 Integral theorems

Gradient theorem:

$$\int_{\mathbf{r}_a}^{\mathbf{r}_b} \nabla \phi \cdot d\mathbf{r} = \phi(\mathbf{r}_b) - \phi(\mathbf{r}_a) \quad (\text{independent of path}). \quad (\text{I.1})$$

Gauss' theorem:

$$\int_V \nabla \cdot \mathbf{F} \, d^3r = \oint_S \mathbf{F} \cdot d\mathbf{S} \quad (S \text{ bounds } V). \quad (\text{I.2})$$

Corollary to Gauss' Theorem:

$$\int_V \nabla \times \mathbf{F} \, d^3r = - \oint_S \mathbf{F} \times d\mathbf{S} \quad (S \text{ bounds } V). \quad (\text{I.3})$$

Stokes' theorem:

$$\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\Gamma} \mathbf{F} \cdot d\mathbf{r} \quad (\text{where } \Gamma \text{ bounds } S). \quad (\text{I.4})$$

## I.2 Product rules

$$\nabla(uv) = u(\nabla v) + v(\nabla u) \quad (\text{I.5})$$

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} \quad (\text{I.6})$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{A} \quad (\text{I.7})$$

$$\nabla \cdot (u\mathbf{A}) = (\nabla u) \cdot \mathbf{A} + u(\nabla \cdot \mathbf{A}) \quad (\text{I.8})$$

$$\nabla \times (u\mathbf{A}) = (\nabla u) \times \mathbf{A} + u(\nabla \times \mathbf{A}) \quad (\text{I.9})$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\nabla \cdot \mathbf{B})\mathbf{A} - (\nabla \cdot \mathbf{A})\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \quad (\text{I.10})$$

### I.3 Second derivatives

$$\nabla \cdot (\nabla \psi) = \nabla^2 \psi \quad (\text{I.11})$$

$$\nabla \times (\nabla \psi) = 0 \quad (\text{I.12})$$

$$\nabla(\nabla \cdot \mathbf{A}) = \nabla \times (\nabla \times \mathbf{A}) + \nabla^2 \mathbf{A} \quad (\text{I.13})$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \quad (\text{I.14})$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \quad (\text{I.15})$$

### I.4 Vector operations in Cartesian coordinates $(x, y, z)$

$$d\mathbf{r} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}} \quad (\text{I.16})$$

$$d^3r = dx dy dz \quad (\text{I.17})$$

$$\nabla f = \frac{\partial f}{\partial x} \hat{\mathbf{x}} + \frac{\partial f}{\partial y} \hat{\mathbf{y}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}} \quad (\text{I.18})$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad (\text{I.19})$$

$$\nabla \times \mathbf{A} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \hat{\mathbf{x}} + \left( \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \hat{\mathbf{y}} + \left( \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \hat{\mathbf{z}} \quad (\text{I.20})$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad (\text{I.21})$$

I.5 Vector operations in spherical polar coordinates  $(r, \theta, \phi)$ 

$$d\mathbf{r} = dr \hat{\mathbf{r}} + r d\theta \hat{\boldsymbol{\theta}} + r \sin \theta d\phi \hat{\boldsymbol{\phi}} \quad (\text{I.22})$$

$$d^3r = r^2 dr \sin \theta d\theta d\phi \quad (\text{I.23})$$

$$\nabla f = \frac{\partial f}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}} \quad (\text{I.24})$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_\theta) + \frac{1}{r \sin \theta} \frac{\partial A_\phi}{\partial \phi} \quad (\text{I.25})$$

$$\begin{aligned} \nabla \times \mathbf{A} = & \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} (\sin \theta A_\phi) - \frac{\partial A_\theta}{\partial \phi} \right] \hat{\mathbf{r}} + \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} (r A_\phi) \right] \hat{\boldsymbol{\theta}} \\ & + \frac{1}{r} \left[ \frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right] \hat{\boldsymbol{\phi}} \end{aligned} \quad (\text{I.26})$$

$$\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \quad (\text{I.27})$$



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I.6 Vector operations in cylindrical coordinates  $(\rho, \phi, z)$ 

$$d\mathbf{r} = d\rho \hat{\boldsymbol{\rho}} + \rho d\phi \hat{\boldsymbol{\phi}} + dz \hat{\mathbf{z}} \quad (\text{I.28})$$

$$d^3r = \rho d\rho d\phi dz \quad (\text{I.29})$$

$$\nabla f = \frac{\partial f}{\partial \rho} \hat{\boldsymbol{\rho}} + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial f}{\partial z} \hat{\mathbf{z}} \quad (\text{I.30})$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \quad (\text{I.31})$$

$$\nabla \times \mathbf{A} = \left( \frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \hat{\boldsymbol{\rho}} + \left( \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \hat{\boldsymbol{\phi}} + \frac{1}{\rho} \left( \frac{\partial}{\partial \rho} (\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi} \right) \hat{\mathbf{z}} \quad (\text{I.32})$$

$$\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \quad (\text{I.33})$$

## J Maxwell's Equations (Static Fields)

Differential form:

<b>Vacuum:</b>	<b>Matter:</b>	
$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$	$\nabla \cdot \mathbf{D} = \rho_f$	(Gauss' law)
$\nabla \cdot \mathbf{B} = 0$	$\nabla \cdot \mathbf{B} = 0$	(no magnetic charge)
$\nabla \times \mathbf{E} = 0$	$\nabla \times \mathbf{E} = 0$	( $\mathbf{E}$ is conservative)
$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$	$\nabla \times \mathbf{H} = \mathbf{J}_f$	(Ampère's law)

(J.1)

Integral form:

<b>Vacuum:</b>	<b>Matter:</b>	
$\oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V \rho d^3r$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho_f d^3r$	(Gauss' law)
$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	(no magnetic charge)
$\oint_\Gamma \mathbf{E} \cdot d\mathbf{r} = 0$	$\oint_\Gamma \mathbf{E} \cdot d\mathbf{r} = 0$	( $\mathbf{E}$ is conservative)
$\oint_\Gamma \mathbf{B} \cdot d\mathbf{r} = \mu_0 \int_S \mathbf{J} \cdot d\mathbf{S}$	$\oint_\Gamma \mathbf{H} \cdot d\mathbf{r} = \int_S \mathbf{J}_f \cdot d\mathbf{S}$	(Ampère's law)

(J.2)

Note: in the 1st row  $S$  bounds  $V$ ; in the 2nd row  $S$  is any closed surface; in the 3rd row  $\Gamma$  is any closed loop; in the 4th row  $\Gamma$  bounds  $S$ .